# Further Maths Notes

Common Mistakes

- ***Read the bold words in the exam!***

- ***Always check data entry***

- *Remember to interpret data with the multipliers specified (e.g. in thousands)*

- *Write equations in terms of variables*

- *Watch for a+bx and ax+b forms*

- *Remember to seasonalise data after predictions*

- *Ensure calculator is in **degrees***

- *Always write true bearings as 90°**T***

- *Always check the feasible region with a point*

- *Never delete an answer*

- *Least squares gradient has S_y on top*

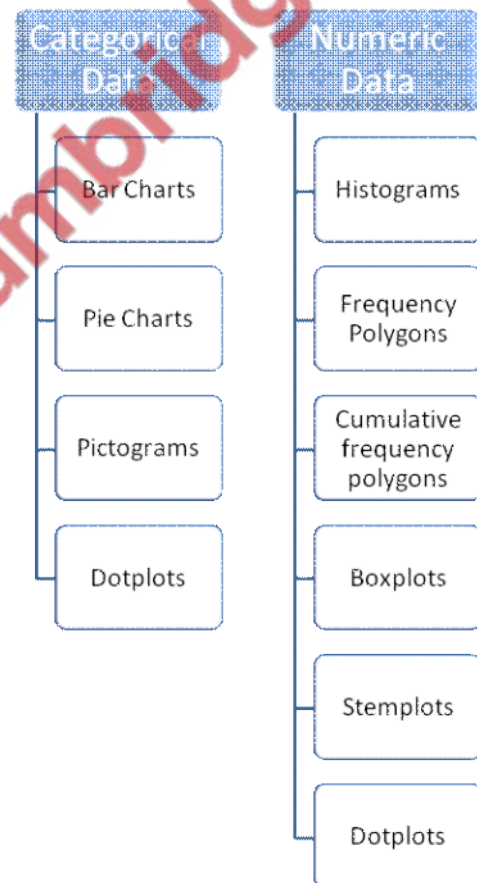- *When crashing, look for vertices common to multiple paths*

Types of Data
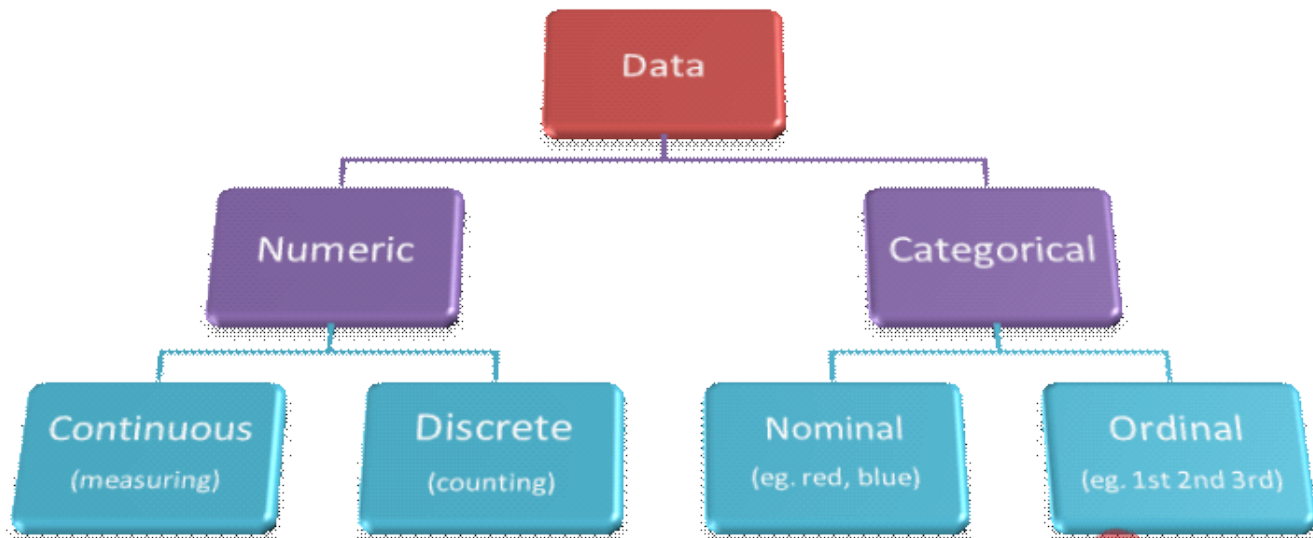
The main techniques used to organise data are:
- Frequency tables
- Dot plots
- Line Plots
- Stem-and-leaf plots

Data is grouped before it is shown in any of the techniques above. There needs to be **more than five** and **no more than fifteen** groups.

When displaying data, remember:
- Title
- Label axes
- Label important points
- Use the key

| Categorical Data	Numeric Data
Bar Charts	Histograms
Pie Charts	Frequency Polygons
Pictograms	Cumulative frequency polygons
Dotplots	Boxplots
	Stemplots
	Dotplots

Skew

Skew determines where data is most spread out. Most distributions are skewed. Skewed data is usually a result of a barrier preventing data from going past a limit.

**Negatively skewed** data has a 'tail' pointing to the left.
This means that:
- ○ Most of the data is concentrated around the **higher** values of x
- ○ Mean, median and mode will not be the same
- ○ Mean, median and mode are located in the body of the curve (the right)
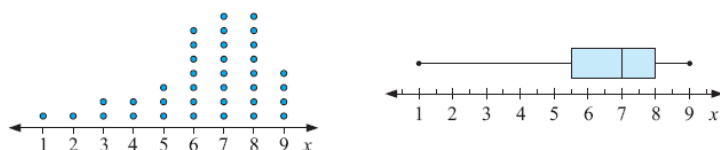
**Positively skewed** data has a 'tail' pointing to the right.
This means that:
- ○ Most of the data is concentrated around the **lower** values of x
- ○ Mean, median and mode will not be the same
- ○ Mean, median and mode are located in the body of the curve (the left)
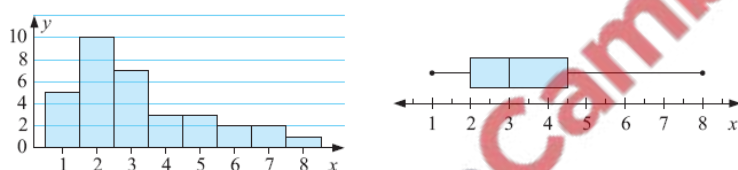
For skewed data, the median and IQR is usually used.

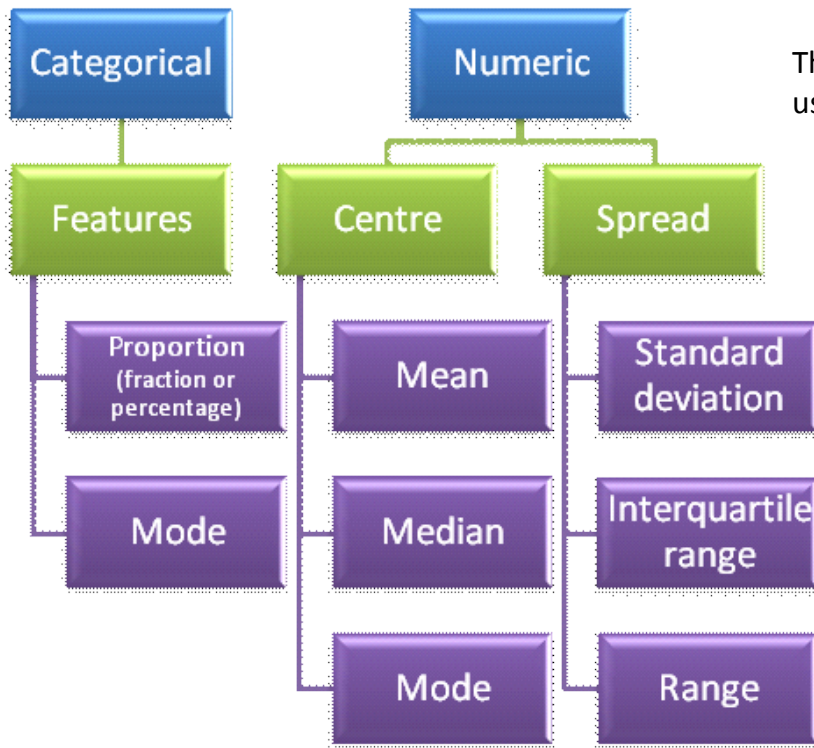**Perfectly symmetrical** data has an **equal** mean, median and mode.
Frequency histograms and polygons show a bell-shaped curve.
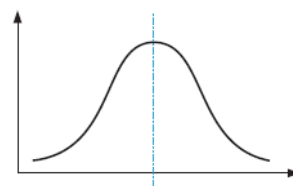
Bimodal data shows two equal modes. This usually indicates that there are two groups of data that need to be separated (eg. height of boys and girls)
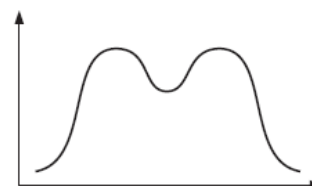
Analysing data

The following summary statistics are usually used together:

- o Mean and standard deviation
- o Median and IQR
- o Mode and range

Statistic	Term number
Q1	$\dfrac{n+1}{4}$
Median	$\dfrac{n+1}{2}$
Q3	$\dfrac{3(n+1)}{4}$

Mode

The mode is the most commonly occurring value. There can be two or more modes.

The mode is **not usually** an accurate measure of centre, and is only used when there is a high number of scores.

Median

The median is the middle value. 50% of data lies either side of the median.

If there are two middle values, it is their average. To calculate the median of grouped data, use a cumulative frequency table.

The median does not take all values into account, and is therefore not affected by extreme values. The median is usually used for skewed distributions. It may not be equal to one of the values in the data set.

Mean

The mean is the average value.

For **ungrouped** data:

$$\bar{x} = \frac{\sum x}{n}$$

For **grouped** data:

$$\bar{x} = \frac{\sum(midpoint \times frequency)}{\sum frequency}$$

The mean is not necessarily equal to a value in the data set.
The mean takes all data into account, and is therefore affected by extreme values.

Range

The range is the difference between the smallest value and the largest. It is influenced by outliers.

For grouped data, the range is the difference between the highest and lowest groups with values.

Interquartile range

The IQR is the range between which 50% of values lie. It is not affected by outliers. It is best obtained using a calculator.

Standard Deviation and Variance

- The variance is the square of the standard deviation. Both values show the distribution of data around the mean.
- Both standard deviation and variance are always positive numbers.
- Both standard deviation and variance are affected by outliers.
- To estimate the standard deviation of a dataset, the following formula is used:

$$s = \frac{range}{4}$$

This formula accounts for 95% of data being within two standard deviations of the mean.

The 68-95-99.7 Rule
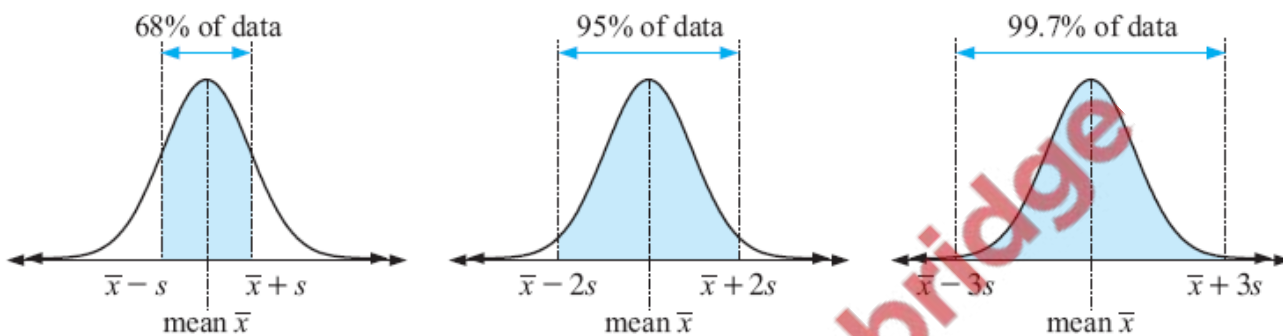
68% of values lie within **one** standard deviation of the mean.
95% of values lie within **two** standard deviations of the mean.
99.7% of values lie within **three** standard deviations of the mean.

Z-Scores

Z-scores are used to compare values between different sets of data. For example, the height of a girl can be compared with the height of a boy by comparing them with their gender and age.
The formula for z-scores is:

$$z = \frac{raw\ score - mean}{standard\ deviation}$$

The following formulas can also be used to determine information:

$$raw\ score = zs + m$$
$$z = \frac{r - m}{s}$$
$$m = -zs + r$$

The z-scores then function like normal data distributions with a mean of zero and a standard deviation of one. The 68-95-99.7 rule can then be applied to the standardised data.

Types of Bivariate Data

Bivariate data sets can be of three types:
- ○ **Categorical** - **Categorical**
- ○ **Numeric** - **Categorical**
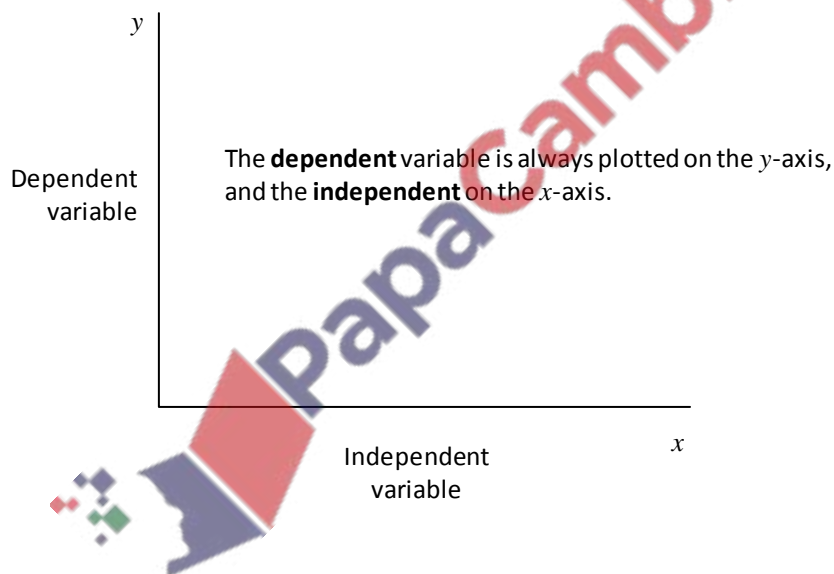- ○ **Numeric** - **Numeric**

For any bivariate data set, one variable is **dependent** and the other **independent**.

*The **dependent** variable responds to change in the **independent** variable.*
*The **independent** variable explains the change in the **dependent** variable.*

The objective of bivariate analysis is to determine whether a relationship exists and if so, its:
- ○ Strength
  - ▪ r values - **perfect**, **strong**, **moderate**, **weak** or **none**
  - ▪ r² values
- ○ Form
  - ▪ **Linear** or **non-linear**
- ○ Direction
  - ▪ **Positive** or **negative**

The **dependent** variable is always plotted on the *y*-axis, and the **independent** on the *x*-axis.

Correlation

1	Strong positive
0.75	Moderate positive
0.5	Weak positive
0.25	No correlation
-0.25	Weak negative
-0.5	Moderate negative
-0.75	Strong negative
-1	

This diagram shows values of Pearson's Product-Moment Correlation Coefficient and what strength relationship they represent.

Positive values always represent relationships with a **positive** gradient, and **negative** values always represent relationships with a **negative** gradient.

Note that:
- It is designed for **linear data only**
- It should be used with caution if **outliers** are present

The Coefficient of Determination is equal to r^2. It describes the **influence** the independent variable has on the dependent variable. It is usually expressed as a percentage. The standard analysis is:

The coefficient of determination calculated to be **[r^2]** shows that **[r^2%]** of the variation in **[dependent variable]** can be explained by variation in **[independent variable]**. The other **[(100-r^2)]%** of variation in **[dependent variable]** can be explained by other factors or influences.
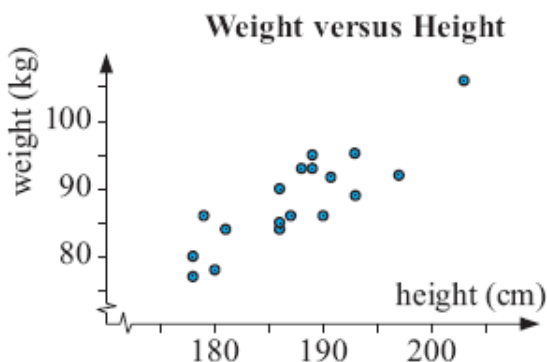
Regressions

Line of best fit

To calculate a line of best fit, first draw a line through the data so that approximately half the data points are on each side. Then, find two points and use the following equations to calculate the equation of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = m(x - x_1)$$

When calculating the equation of the line of best fit, it is sometimes possible to estimate the y-intercept, however before doing so ensure that both axes start at zero.

Weight versus Height

In this graph, it is easy to estimate the y-intercept to be 60, however the axes are broken so that would be incorrect.

Three Median Regression

The three-median regression line is formed as follows:

1. Plot the data on a scatterplot
2. Divide the points symmetrically into three groups
3. Find the median point of each group by finding the median of the x and y values
4. Use the slope of the outside points to draw a line that is **one third** of the way towards the middle point.

To calculate the equation of the three-median regression line, use the following formulae:

$$m = \frac{y_R - y_L}{x_R - x_L}$$

$$c = \frac{[(y_L + y_M + y_U) - m(x_L + x_M + x_U)]}{3}$$

Least Squares Regression

The least squares regression line is formed using the following formulae:

$$m = \frac{rS_y}{S_x} \qquad c = \bar{y} - m\bar{x} \qquad r = \frac{\Sigma(Z_x Z_y)}{n - 1}$$

The **Three-Median** Regression line is usually used for data **with outliers**, and the **Least Squares** Regression line is usually used for data **without outliers**.
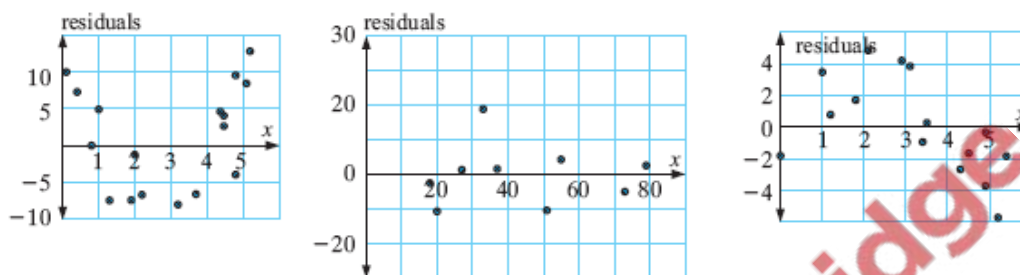
Residuals

The residual for a data point is its vertical distance from a regression line. It is calculated using the following formula:

$$residual = actual\ value - predicted\ value$$
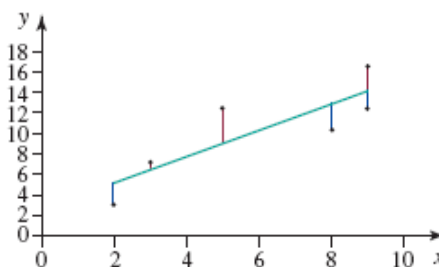
A residual plot (shown below) shows important information about a relationship.

A residual plot with a **valley** (left) or a **crest** (right) indicates that data is **non-linear** and a transformation should be applied to the data.

A residual plot with a **random pattern** (centre) indicates that data is **linear**.
WARNING: Residual plots are never 'scattered', they are always **random**.

Transformations

The following tables shows which transformations can be used on certain scatterplot shapes, and the effect of certain transformations.

Curve	Transformations to test
	$x^2,$ $\log y,$ or $\dfrac{1}{y}$
	$y^2,$ $\log x,$ or $\dfrac{1}{x}$
	$\log x,$ $\log y,$ $\dfrac{1}{x},$ or $\dfrac{1}{y}$
	x^2 or y^2

When a transformation is needed, use the values of r and r^2 to help determine which is best.

Summarising Time Series Data

Time series data is simply data with a timeframe as the independent variable.
There are four ways in which it can be described:

1. **Secular Trend**
   Data displays a trend (or secular trend) when a consistent increase or decrease can be seen in the data over a significant period of time. A trend line can be fitted to such data.
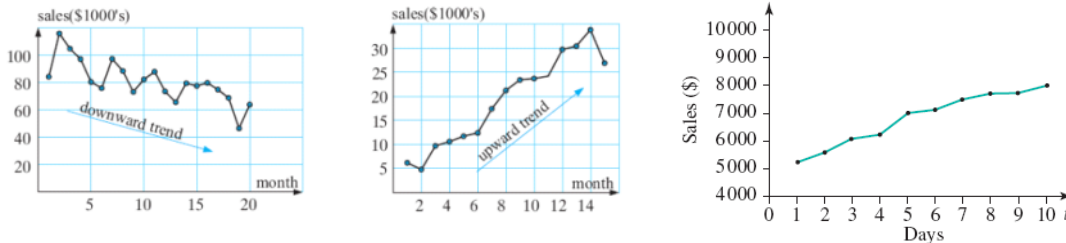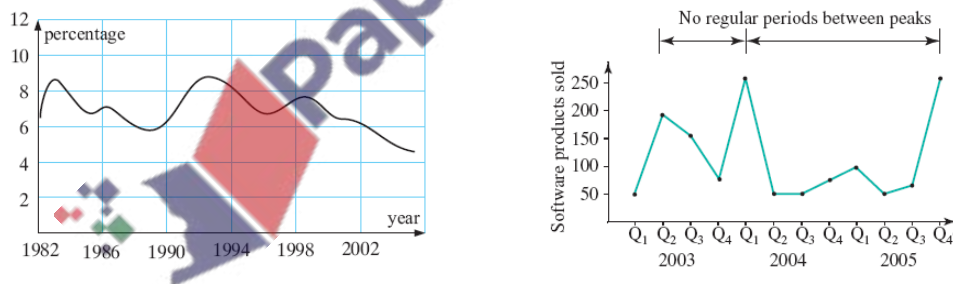

2. **Seasonality**
   Seasonal data changes or fluctuates at given intervals, with given intensities. For example, sales of warm drinks might fluctuate in winter. This data can be deseasonalised.
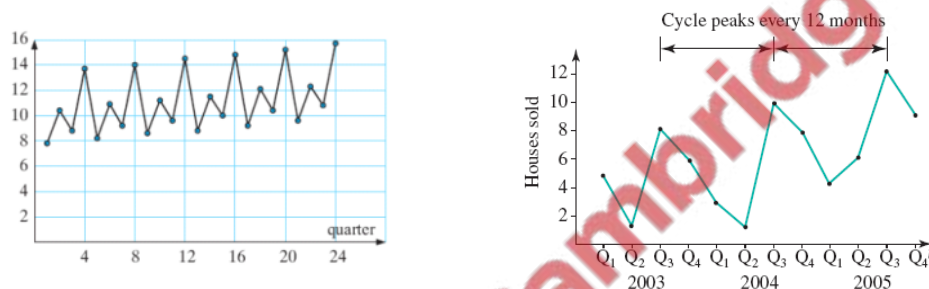
3. **Cyclic**
   Cyclic data shows fluctuations, but not at consistent intervals, amplitudes or seasons. This includes data such as stock prices.

4. **Random**
   Random data shows no pattern. All fluctuations occur by chance and cannot be predicted.
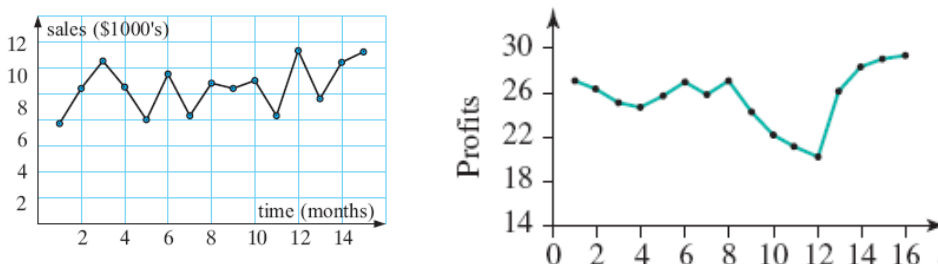
Smoothing Time Series Data

Time series data can be smoothed in two ways - moving average smoothing and median smoothing.

Moving Average Smoothing

Moving average smoothing is simple. Each smoothed data point is equal to the average of a specified number of points surrounding it. For example, if a five point moving average is chosen, then each consecutive group of five points is averaged to give the points in the smoothed data set.

Median Smoothing

Median smoothing is very similar to moving average smoothing, however the median of the points is taken instead of the average. This method is generally preferred to moving average smoothing when outliers are present.

The number of points to smooth to is generally equal to the number of data points in a time cycle. For example, five points would be used for data from each working week, and twelve points would be used for monthly data.

Deseasonalising Data

A procedure exists for deseasonalising data. It involves the following formulae:

$$seasonal\ average = average(actual\ data\ for\ one\ season)$$

$$seasonal\ proportion = \frac{actual\ figure}{seasonal\ average}$$

$$seasonal\ index = \frac{original\ figure}{deseasonalised\ figure} = average(seasonal\ proportions)$$

$$actual\ figure = deseasonalised\ figure \times seasonal\ index$$

$$deseasonalised\ figure = \frac{original\ figure}{seasonal\ index}$$

Seasonal indices can also be used to comment on the relationship between seasons, for example:

- A seasonal index of **1.3** means that the season is **30% above** the average of the seasons
- A seasonal index of **1.0** means that the season is **equal to** the average of the seasons
- A seasonal index of **0.8** means that the season is **20% below** the average of the seasons

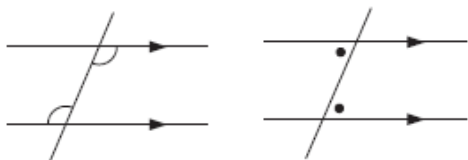**The sum of the seasonal indices is always equal to the number of seasons.**

Parallel Lines and Triangles
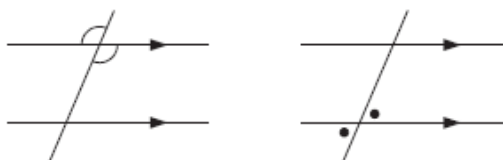
Parallel lines have the following properties:
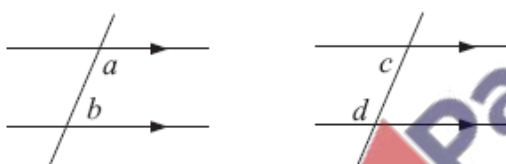
1. Corresponding angles are equal
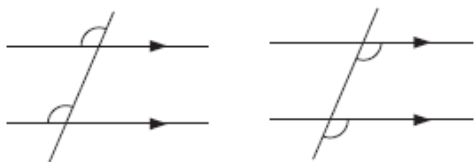
2. Alternate angles are equal
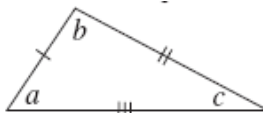
3. Vertically opposite angles are equal

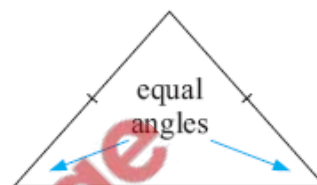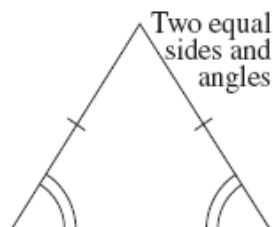**4. Co-interior angles add to 180°**

Triangles have the following properties:

1. **In a scalene triangle, all angles and sides are different.**

2. **In an isosceles triangle, two angles are the same and two side lengths are the same. The angles opposite each of the equal sides are the same.**
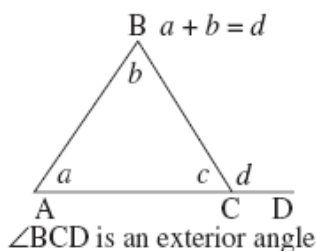
Two equal sides and angles

equal angles

3. **In an equilateral triangle, all sides are the same and all angles are 60°.**

All equal sides and angles
60° 60° 60°
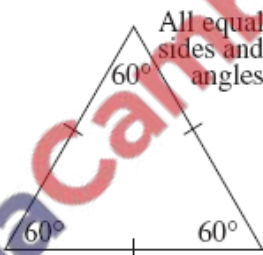
4. **Angles in every triangle add to 180°.**

5. **Corresponding interior and exterior angles always add to 180°. The sum of the angles opposite an angle in a triangle are always equal to the corresponding exterior angle ($a+b=d$)**

B $a + b = d$
b
a c d
A   C   D
$\angle BCD$ is an exterior angle

Regular Polygons

In a regular polygon, all **sides** and **angles** are **equal**.

Exterior angle

interior angles $\left(\frac{360}{n}\right)^\circ$
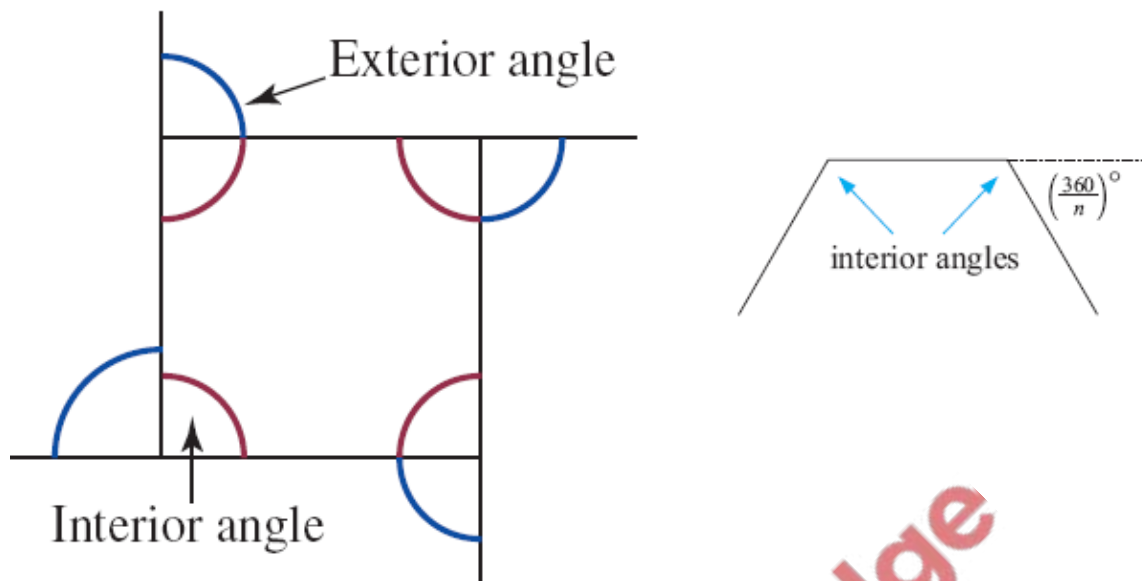
Interior angle

The **sum** of the **interior** angles of an n-sided polygon is:	The **sum** of the **exterior** angles of **any** polygon is:
$s = (180n - 360)^\circ$	$s = 360^\circ$

Size of each **interior** angle:	Size of each **exterior** angle:
$s = \left(\dfrac{360}{n}\right)^\circ$	$s = \left(180 - \dfrac{360}{n}\right)^\circ$

Right Angled Triangles
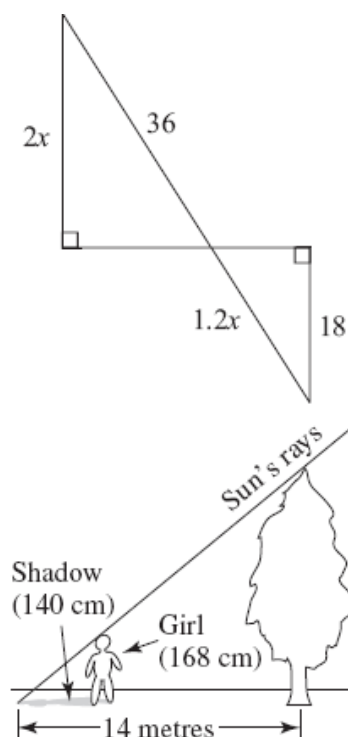

c (or Hypotenuse)

a

b
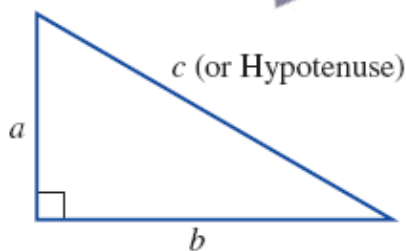
The Pythagorean Theorem states that:

$$a^2 + b^2 = c^2$$

$$a = \sqrt{c^2 - b^2}$$

Note that c always refers to the hypotenuse, however exams may refer to the sides differently.

Similarity and Ratios

Two figures are **similar** if they are the **same shape** and the corresponding dimensions are in the **same ratio**.

Two triangles are similar if the **angles** are **equal**.

The sides of one similar shape are all **proportional** to the sides of the other, so ratios and fractions can be used to calculate the unknown lengths.

Ratios between areas and volumes can be calculated from the length ratio. The following formulae are used:

Length:	$a:b$
Area:	$a^2:b^2$
Volume:	$a^3:b^3$

Area:	$a:b$
Length:	$\sqrt{a}:\sqrt{b}$
Volume:	$a\sqrt{a}:b\sqrt{b}$

Volume:	$a:b$
Area:	$a^{\frac{2}{3}}:b^{\frac{2}{3}}$
Length:	$\sqrt[3]{a}:\sqrt[3]{b}$

Ratios given in questions, unless otherwise stated, are **always length ratios**.

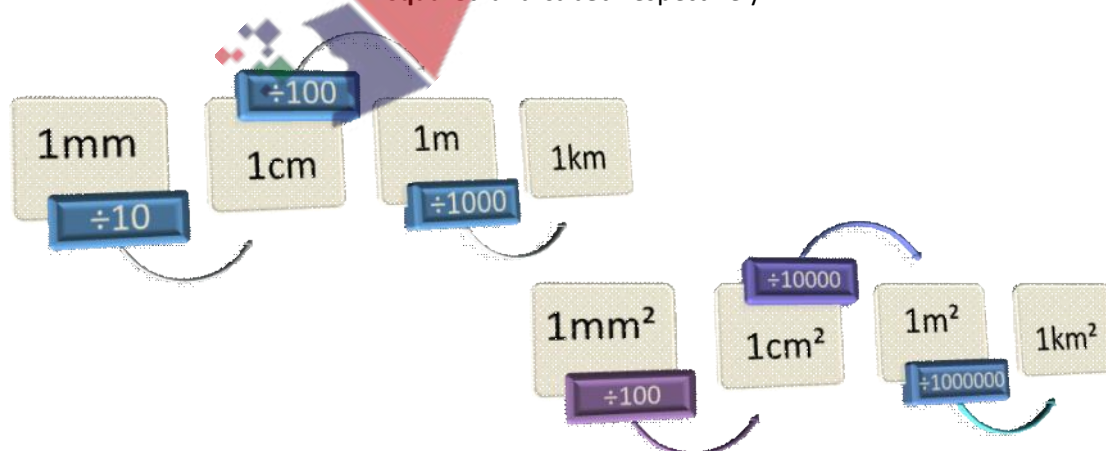Remember, the number of dimensions being **scaled** matches the number of dimensions in the **ratio**.

Example:
$ratio = 140:1400 = 1:10$
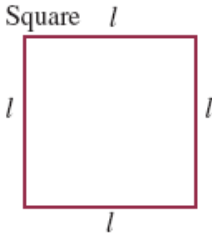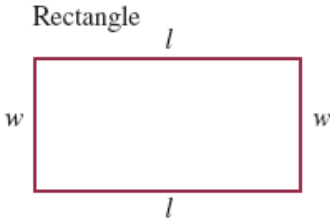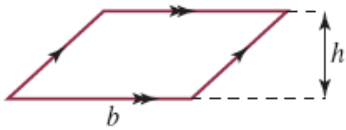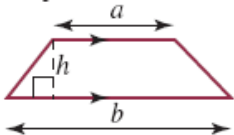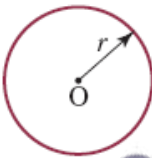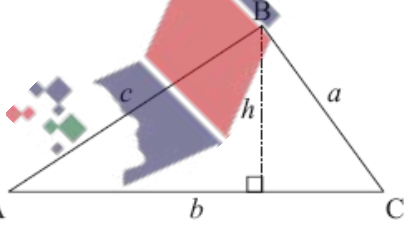$\dfrac{1}{10} = \dfrac{x}{168} \quad \therefore x = 1680cm = 16.8m$

Converting Units

Ratios are used when converting units. Remember that when **area** and **volume** is converted, the ratios are **squared** and **cubed** respectively.
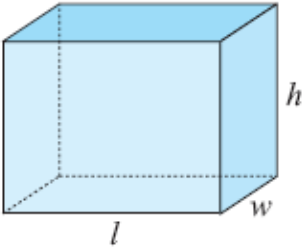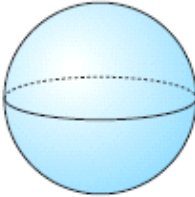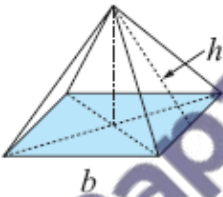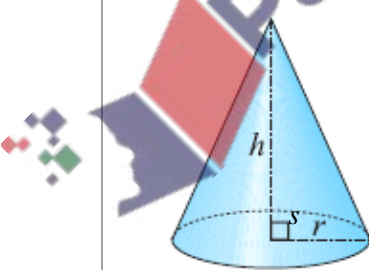
Area of shapes

The following table shows the formulae used for calculating the area of simple shapes:

Name	Diagram	Formulae used
Square	Square l	$A = l^2$
Rectangle	Rectangle l	$A = l \times w$
Parallelogram		$A = b \times h$
Trapezium		$A = \frac{1}{2}(a + b) \times h$
Circle		$A = \pi r^2$
Triangle		$A = \frac{1}{2} \times b \times h$ $Area = \frac{1}{2} b \times c \times \sin A$ *(A is the included angle)* $A = \sqrt{s(s-a)(s-b)(s-c)}, s = \dfrac{a+b+c}{2}$

Volume and TSA

The following table shows formulae used to calculate the volume and total surface area of simple shapes:

| Name	Diagram	Volume	Total Surface Area
Cuboid		$V = l \times w \times h$	$TSA = 2lw + 2wh + 2lh$
Sphere		$V = \frac{4}{3}\pi r^3$	$TSA = 4\pi r^2$
Cylinder		$V = \pi r^2 h$	$TSA = 2\pi rh + 2\pi r^2$
Right square pyramid		$V = \frac{1}{3}b^2 h$	$TSA = b^2 + 2bh$
Cone		$V = \frac{1}{3}\pi r^2 h$	$TSA = \pi r^2 + \pi rs$ $TSA = r^2\pi + r\pi\sqrt{h^2 + r^2}$

Note that for rectangular pyramids, the following formula can be used to determine the height:

$$h = \frac{\sqrt{4s^2 - w^2 - l^2}}{2}$$

h=height, w=width, s=slant height, l=length

Trigonometry

Right Angled Triangles
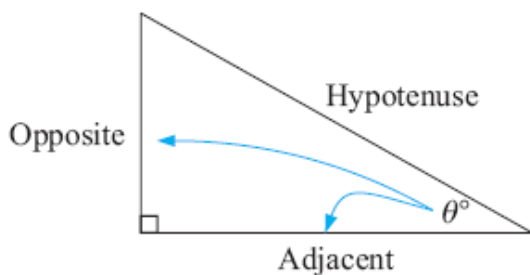
$$\sin \theta = \frac{opposite}{hypotenuse}$$

$$\cos \theta = \frac{adjacent}{hypotenuse}$$

$$\tan \theta = \frac{opposite}{adjacent}$$

The Sine Rule (all triangles)
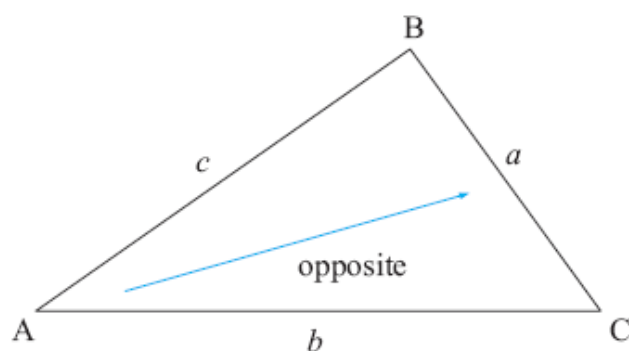
The sine rule is used with two sides and two corresponding angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Cosine Rule (all triangles)

The cosine rule is used with three sides and one angle.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Bearings and Contours

N
clockwise

Town B

228°
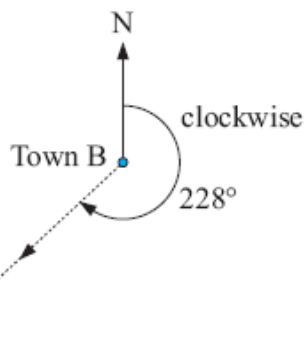
A bearing is a way of specifying **direction**
from one place to another.

There are two types of bearings:
- Compass bearings (e.g. S48°W)
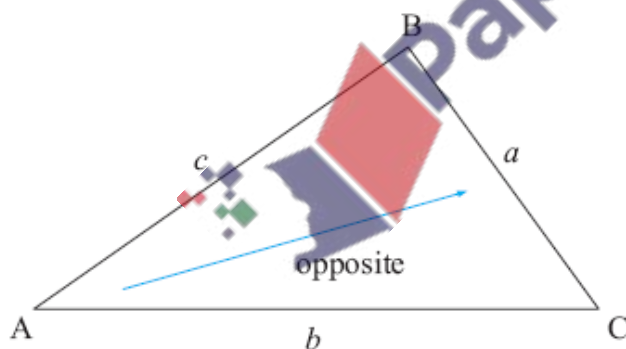- True bearings (e.g. 228°T)
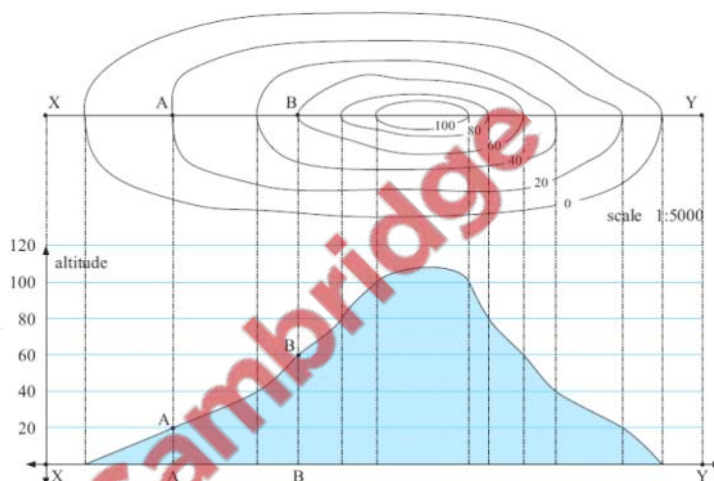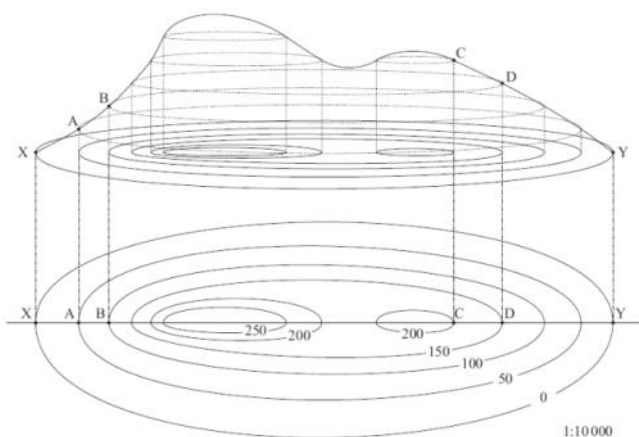
Contour maps graphically show **heights**. Each line on a contour map
represents the area on the map where land is **exactly** that height.

Dense lines show a **steep** slope, and lines that
are **spaced apart** show a **flatter** slope.

The slope can be calculated simply by using the
rise-over-run formula.

Types of Networks

Graph type	Number of edges with n vertices
Complete	$\dfrac{n(n-1)}{2}$
Connected	$n-1$

Degenerate graph   Complete graph

Connected graph   Disconnected graph   Simple graph

Non-simple graphs   Subgraph   Bipartite graph

Planar Graphs and Euler's Formula

A **planar** graph is one where **edges do not cross**. Some graphs can be redrawn to be planar, others not. Euler's formula is used to confirm whether graphs are planar or not.

$$v + f - e = 2$$

e	Edges
v	Vertices
f	Regions or faces

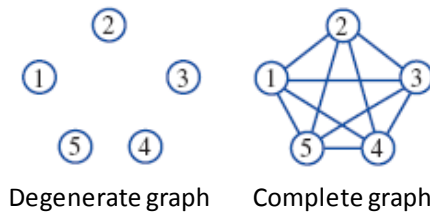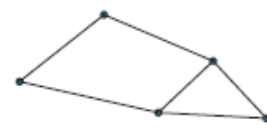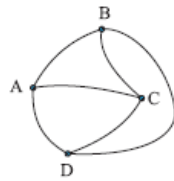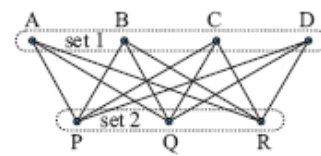**Note:** the area outside the graph is counted as one region.

Hint: if v is less than or equal to four, then the graph is planar.
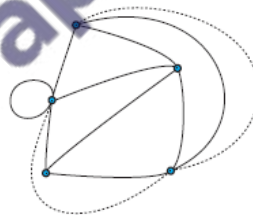
Original graph   Redrawn graph

v	4
e	8
f	6

4+6-8=2

Euler's formula

Matrix Representation

Networks can be represented using adjacency matrices. The numbers on the leading diagonal represent loops, and all undirected graphs are symmetrical about the leading diagonal.

From

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{cccccc} & \multicolumn{5}{c}{To} \\ & A & B & C & D & E \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$
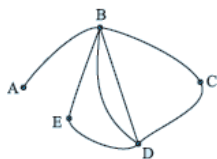
Loops

A loop in an undirected network adds **two** to the degree of a vertex, and adds **one** to the leading diagonal of a matrix. For example:

Node A has degree 3.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{ccccc} A & B & C & D \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

Paths, Circuits and Trees

Paths

A path is a sequence of steps between adjacent nodes. In this graph, B-D-A-C and C-A-B-D are two of the paths which could exist. B-C-D-A could not exist because B is not adjacent to C.

Circuits

A circuit is a sequence of steps between adjacent nodes which starts and finishes at the same vertex. In this graph, B-D-C-A-B and A-D-B-A-D-C-A are two of the circuits which could exist.

Eulerian paths and circuits

Eulerian paths and circuits pass each **edge** only once. Eulerian paths exist where there are two or less vertices with odd degree. Eulerian circuits exist when al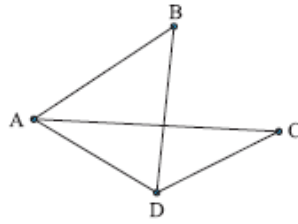l vertices have even degree. On the graph above, D-A-C-D-B-A represents an Eulerian path, and no Eulerian circuits exist.

Hamiltonian paths and circuits

Hamiltonian paths and circuits are those which pass through each **vertex** only once. Examples in the above graph include B-A-D-C and D-C-A-B-D.

Trees

A **tree** is a connected, simple graph with no circuits. A **spanning tree** is a subgraph of a connected graph which contains **all the vertices** of the original graph. The weight of a spanning tree is the combined weight of all its edges, and there are two ways in which the minimum-weight spanning tree can be found:

- **The nearest neighbour algorithm** in which the longest edge, which doesn't disconnect the graph, is removed. This is repeated until no more edges can be removed.
- **Prim's algorithm** which involves choosing random vertex as a starting graph and constantly building to it by adding the shortest edges which will connect it to another node.

The shortest path

The shortest path algorithm (or Dijkstra's Algorithm) is used to find the shortest path between two nodes. It involves working from start to finish, labeling each vertex with the shortest path between it and the starting node.

The Chinese Postman Problem

The Chinese Postman Problem involves finding the shortest circuit (route) along a network (town) which covers all edges (roads) and starts and finishes at the same spot (the Post Office). It must be an Eulerian circuit if possible, otherwise the routes between the odd-degree vertices must be travelled twice.

The Travelling Salesman Problem

The Travelling Salesman Problem (TSP) involves finding the shortest circuit (route) which covers all vertices (houses) in a network (town) in the minimal distance. It must be a Hamiltonian circuit if possible, but there is no systematic way of solving this problem.

Directed Graphs

In a directed graph, each edge has a direction. Also, each vertex in a network can be reachable or unreachable.
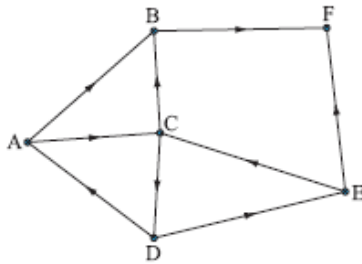
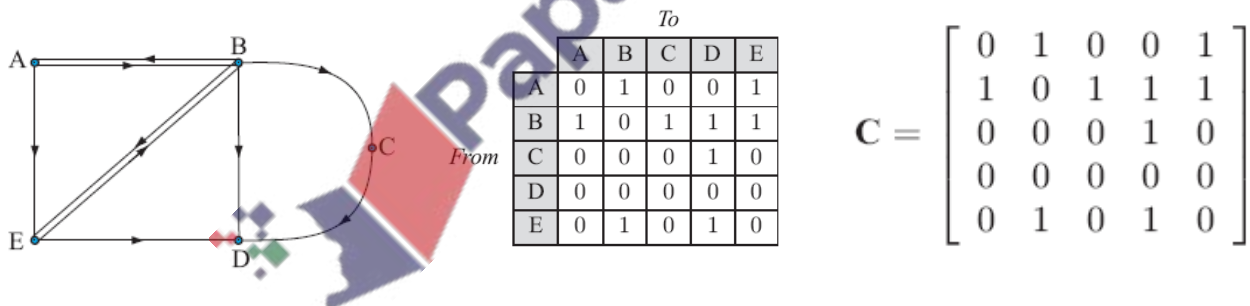In this network, no node is reachable from F, and A is not reachable from B.

Network Flow

Weighted graphs can be used to model the flow of people, water or traffic. The flow is always from the *source* vertex to the *sink* vertex. The weight of an edge represents its capacity.

Cuts are used as a way of preventing all flow from source to sink. A valid cut must completely isolate the source from the sink. By adding the weights of the cut edges, the value of a cut can be obtained. The minimum value cut that can be made represents the maximum flow possible through the network.

If a cut passes through an edge which flows from **sink to source**, it is called a *double cut*. The weight of a double cut is equal to the sum of all of its edges which flow from **source to sink**.

Connectivity Matrices

Every directed network can be represented in a matrix form, like undirected networks. The matrices that represent directed graphs however, are not always symmetrical and can tell us more about a network.

	To				
From	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	1
C	0	0	0	1	0
D	0	0	0	0	0
E	0	1	0	1	0

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The edges in the network above are represented in the table, which can be translated directly into a matrix. This matrix shows all of the one-stage paths between any two vertices. If this matrix is squared, the resulting matrix shows the two-step pathways between nodes. The original and the squared matrix can be added to find the winning team in a round-robin tournament.

Project Management
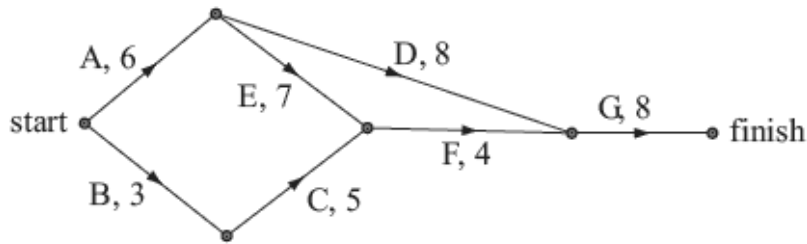
The network above shows a plan for a project. It could be anything from building a house to getting ready in the morning. The critical path represents the longest path from the start to the finish, and hence the path on which the finishing time depends. The critical path can be found using forward and backward scanning.
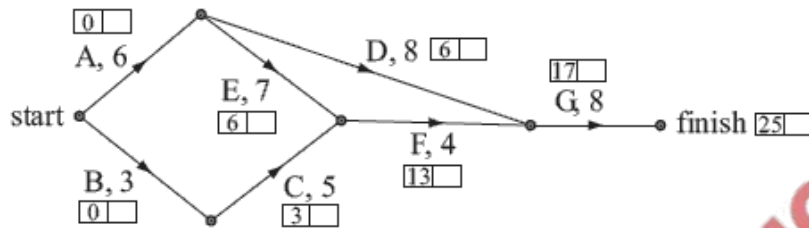
Forward scanning involves finding the **earliest start time** for every task. Starting at zero, the longest path from the start to each node (the start of each task) is calculated and written in the left-hand box. The number written in the last box is the critical time.
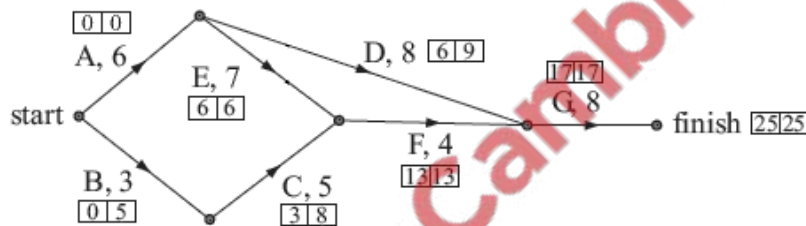

Backward scanning involves finding the longest path from the finish vertex to each node (the start of each task). This gives us the **latest start time** for each activity, before it delays the finishing time of the project. The slack (or float) time, in which nothing is being done on a particular path, is calculated by **LST - EST**. The critical path can be found by looking for activities with a float time of zero.

Crashing a project

If a project looks like running overtime, it may be crashed. Crashing involves spending extra money to reduce the time taken by certain activities in order to avoid costs of completing the project late. The following steps illustrate the crashing process:

One. Write down all of the paths from the start node to the finish node, and determine the length of each.

Two. Calculate the cost per day of crashing each activity (note that some questions may already have the cost per day, and watch for reductions that must be made in full)

Three. Reduce the cheapest (cost per day) activity **on the critical path** by one day.

Four. Calculate the new lengths of all of the paths.

Five. Repeat steps three and four, each time using the new longest path after reductions. Stop when the budget is reached or the longest path cannot be reduced. The new critical path is the longest path.

Assignment Problems
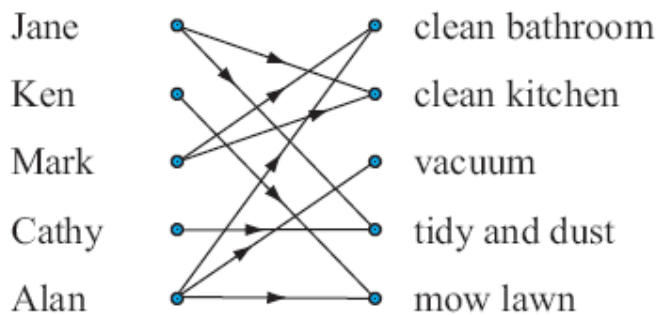
Assignment problems can be represented using bipartite graphs like the one opposite. Sometimes the edges are weighted, in which case the Hungarian Algorithm can be used to find the optimal combination.

The Hungarian Algorithm

The following steps find the minimum-weight assignment of tasks in a bipartite graph:

One. Represent the data in a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} x \ \ y \ \ z \\ \left[\begin{array}{ccc} 8 & 4 & 5 \\ 6 & 4 & 6 \\ 7 & 5 & 4 \end{array} \right] \end{array}$$

Two. Check to see if an assignment can be easily made.

Three. Find the smallest number in each **row** and subtract it from every number in that row.

$$\left[\begin{array}{ccc} 4 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{array} \right]$$

Four. Check to see if an assignment can be made.

Five. Find the smallest number in each **column** and subtract it from every number in that column.

$$\left[\begin{array}{ccc} 2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{array} \right]$$

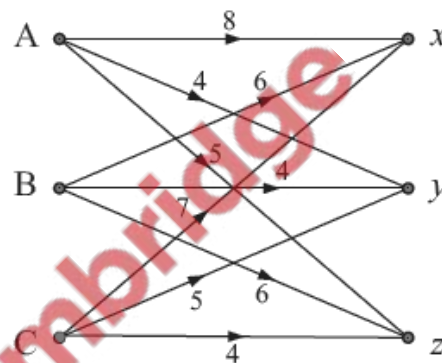Six. Check to see if an assignment can be made.

Seven. Cover all zeroes in the matrix with as few vertical or horizontal lines as possible. Find the minimum uncovered value and:

  a. Add it to every value covered by two lines

  b. Subtract it from every uncovered value

$$\left[\begin{array}{ccc} 2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{array} \right]$$

Eight. Use the zeroes to allocate tasks.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} x \ \ y \ \ z \\ \left[\begin{array}{ccc} 2 & ⓪ & 1 \\ ⓪ & 0 & 2 \\ 1 & 1 & ⓪ \end{array} \right] \end{array} \qquad \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} x \ \ y \ \ z \\ \left[\begin{array}{ccc} 8 & ④ & 5 \\ ⑥ & 4 & 6 \\ 7 & 5 & ④ \end{array} \right] \end{array}$$

Linear Basics

The following formulae are used to calculate the gradient of a straight line:

$$m = \frac{rise}{run} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient of a straight line is always the coefficient of x. Lines with the same gradient are parallel.
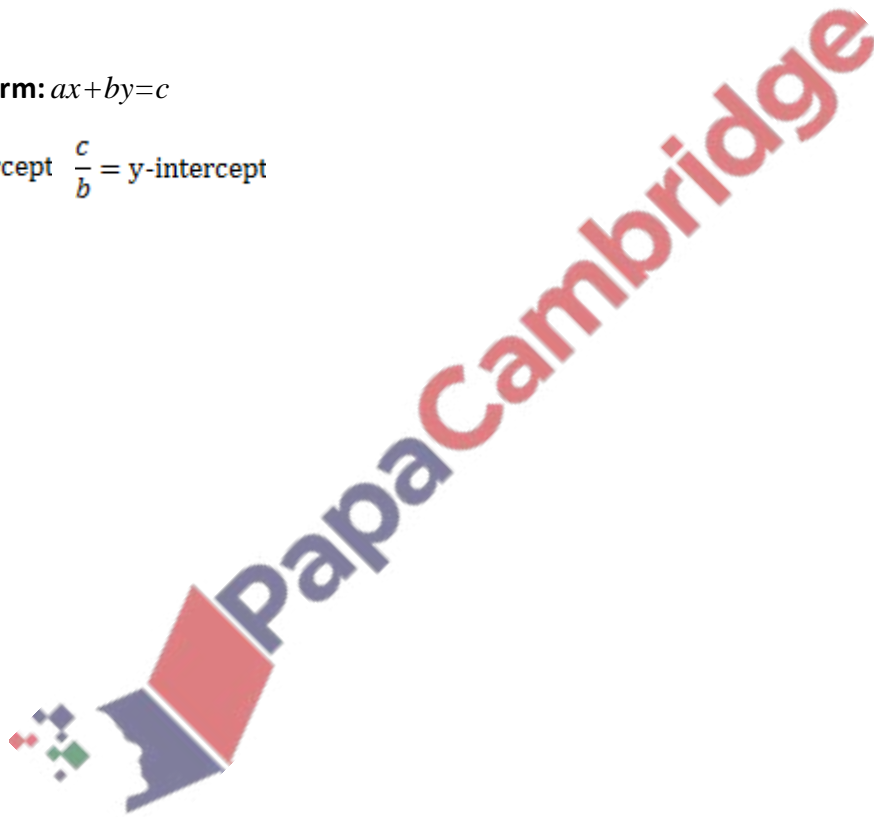
Equation Form

Equations can be presented in one of two forms:

Gradient-intercept form: $y = ax + b$

$a = \text{gradient} \qquad b = \text{y-intercept}$

General form: $ax + by = c$

$\dfrac{c}{a} = \text{x-intercept} \qquad \dfrac{c}{b} = \text{y-intercept}$

Applications

Simultaneous Equations

Example:
At the footy, three pies and five buckets of chips costs $16.50.
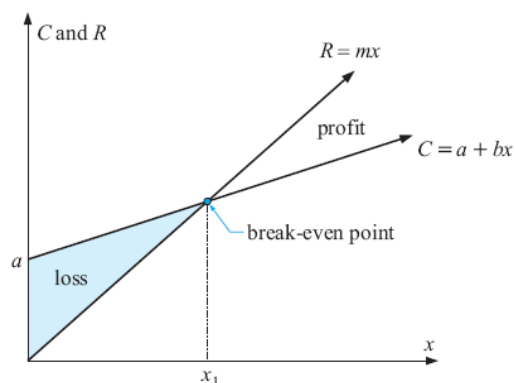Also, 4 pies and 3 buckets of chips costs $15.40.

$3p+5c=16.5$
$4p+3c=15.6$

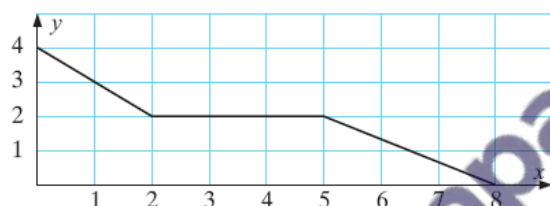*What is the cost of pies and chips?*
Using a calculator, pies cost $2.50 and chips $1.80.

Break-Even Analysis

The break-even point is where cost equals revenue. On the graph opposite, this is represented as x_1. Profit and loss is defined by the difference between this value and the actual number of products sold.

Line Segment Graphs

Line segment graphs occur when more than one equation exists for a graph. For example, cyclists might ride at different speeds for different stages of a journey. Their distance from a given point would be represented by a line segment graph.

Step Graphs
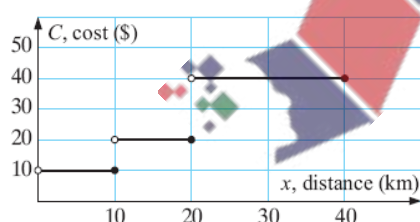
Step graphs represent relationships defined by a series of zero-gradient equations, in situations like parking costs and postage costs.

Non-linear Graphs
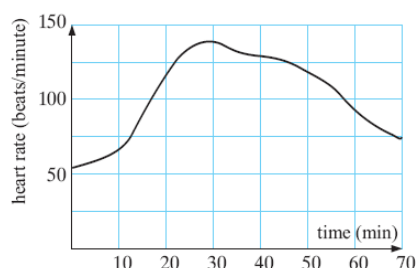
Questions may ask for interpretations of non-linear graphs. Key knowledge includes:
- ○ The maximum point
- ○ The minimum point
- ○ Large and small rates of change

Exponential Graphs

$$y = kx^n$$

This is the general form for an exponential graph. It can be obtained from a set of data by using the *power* regression on the calculator.

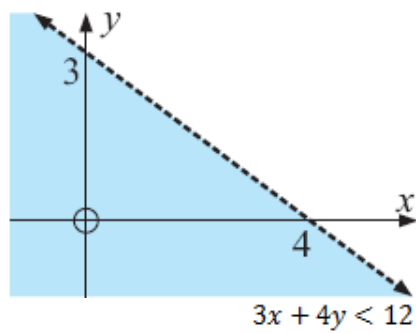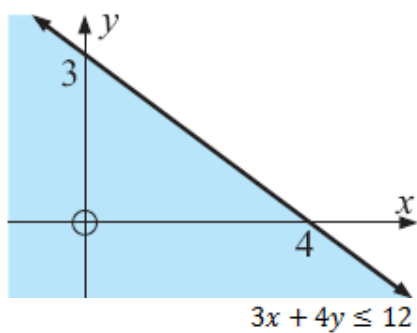The following graphs show the shape of relationships with different values of n.

NEGATIVE

Value of n	Graph
-2	
-1	

POSITIVE

Value of n	Graph
1	
2	
3	

Inequalities

An inequality differs from an equation because it defines a region of a graph, as opposed to a line.

$3x + 4y \leq 12$

$3x + 4y < 12$

If an inequality is in either *general form* or *gradient-intercept form* then the '**<**' sign will refer to the region **below** the boundary and the '**>**' sign will refer to the region **above** the boundary.

Always remember to **use a key** and **dot the line** if it is not included.
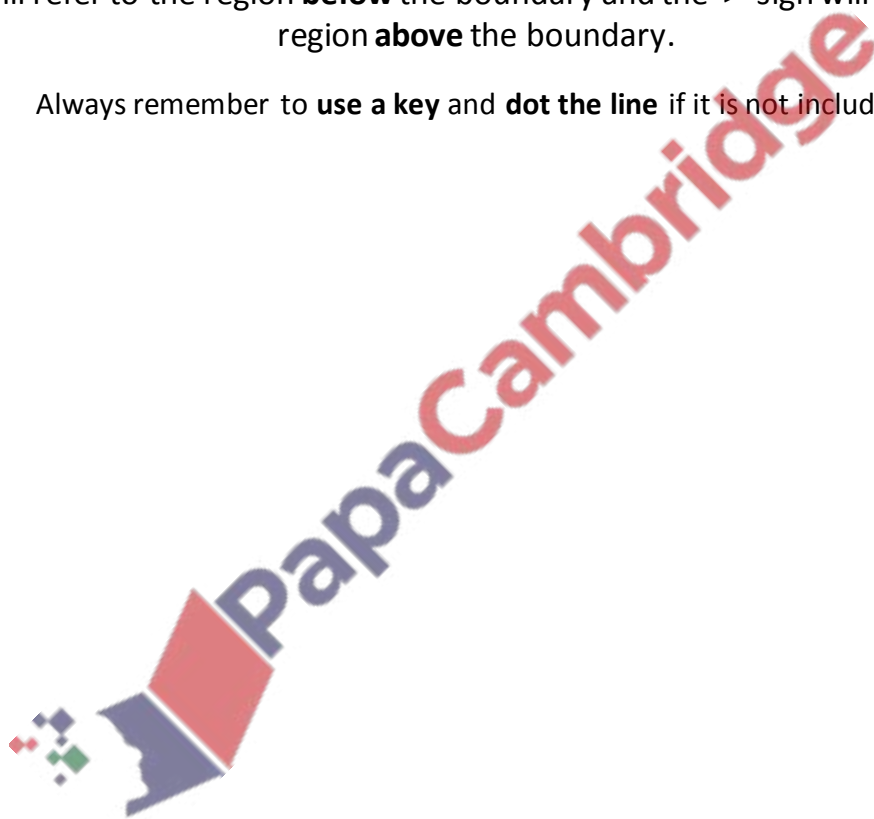
Modeling Inequalities

Inequalities can be used to model situations. To model a situation, a number of constraints are used. Sometimes information can be represented in a table.

Example:

Fred can buy sausages and hamburgers for a BBQ dinner. He cannot buy more than twelve items.
Let x be sausages and y be hamburgers.

$x \geq 0$

$y \geq 0$

$x + y \leq 12$

Sausages cost $2 each and hamburgers $1. He cannot spend more than $15.

$2x + y \leq 15$

To graph a set of inequalities:
  a. Change each inequality to an equation by replacing the inequality with an equals sign.
  b. Plot the lines, dotting those where the line itself is not included,
  c. Shade the feasible region and write a key.
  d. Check that a point within the feasible region fits all the inequalities.

Problem Solving

To solve a linear programming problem, an *objective function* is required. This objective function calculates the cost given the values of x and y. The following steps show how to use the values if the extreme points to solve a linear programming problem:

  One. Find the values of the extreme points by getting the intercepts between the lines. **Make sure each of the extreme points chosen is inside the feasible region.**
  Two. Write the x and y values of these intercepts in a table.
  Three. Make a new column in the table and write the result of the objective function for the x and y values in that row.
  Four. Look for the lowest or highest (depending on the question) value for the objective function. **If there a two lowest or highest values, the ideal values of x and y are not only those listed, but also every value on the line joining those points.**

The Sliding Ruler Method

The Sliding Ruler method is an alternative to the method above. Its procedure is:

  One. Give the objective function an equal value to change it into an equation.
  Two. Plot it on a graph of the feasible region.
  Three. Slide a clear ruler along the page, keeping the same gradient, until the last point in the feasible region is reached. The direction to slide the ruler depends on whether the objective function is being minimised or maximised.
  Four. The last point that is reached in the feasible region is equal to the ideal x and y values for the problem.
  If the ruler finishes on a line, all the points on that line are ideal values.

Proportional Inequality Statements

The following are some statements used in linear programming questions, and their associated inequalities:

For each x, there are at least two y

$$2y \leq x$$

For every eight x, there are at least ten y

$$8x \geq 10y$$

There are at least twice as many x as y

$$2y \leq x$$