Formulas :-

Algebra :-

"Absolute Functions" or "Modulus"

① $|2x+5| = 11$

- $2x+5 = 11$

  $x = 3$

- $2x+5 = -11$

  $x = -8$

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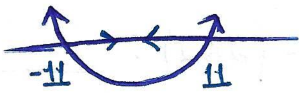
② $|2x+5| < 11$



- $2x+5 > -11$

  $2x > -16$

  $x > -8$

- $2x+5 < 11$

  $2x < 6$

  $x < 3$

hence $\boxed{-8 < x < 3}$

③ $|2x+5| > 7$

- $2x+5 > 7$

  $2x > 2$

  $x > 1$

- $2x+5 < -7$

  $2x < -12$

  $x < -6$

hence $\boxed{1 < x < -6}$

---

④ If x involves on both sides, then squaring both sides :-

$|x-2| < 3-2x$

squaring

$x^2-4x+4 < 9-12x+4x^2$

$\rightarrow -3x^2+8x-5 < 0$

must be ⊕ $\quad 3x^2-8x+5 > 0$

$3x^2 - 5x - 3x + 5 > 0$

$x(3x-5) - 1(3x-5) > 0$

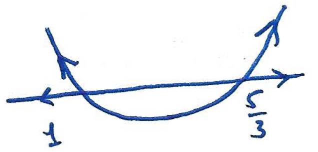
$(x-1)(3x-5) > 0$

↙ $x-1=0$

$x=1$

↘ $3x-5=0$

$x = \frac{5}{3}$



$\boxed{x<1 \quad \text{and} \quad x>\frac{5}{3}}$

<u>Must check</u>

→ $x<1$

eg. $x=0$

$|0-2| < 3-(2\times0)$

$2<3$

correct.

→ $x > \frac{5}{3}$

eg. $x=2$

$|2-2| < 3-(2\times2)$

$0 < -1$

wrong.

<u>hence</u> $\boxed{x<1}$ <u>only</u>

<u>Note</u> $(ax-b)$ → if factor then root is "$+\frac{b}{a}$"

$(x-a)$ → if factor then root is "a".

<u>eg.</u> $(x+1)$ → factor → -1 → root.

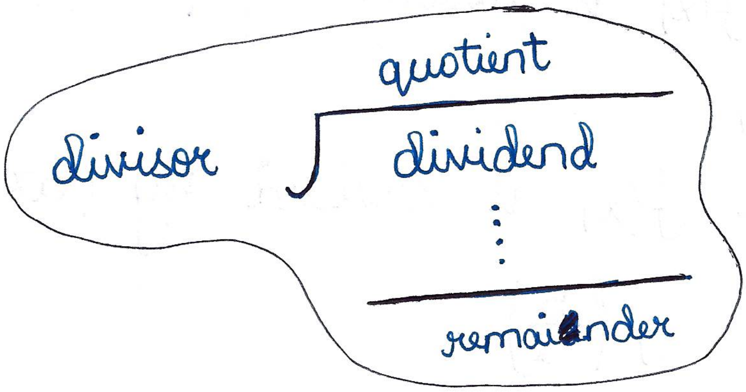
$(x-12)$ → factor → $+12$ → root.

$(2x+10)$ → factor → -5 → root.

$(5x-15)$ → factor → $+3$ → root.

Long Division :-

- $(x^3-2x^2+5x+6) \overset{\text{divided BY}}{\boxed{\div}} (x+2) \underset{\text{same}}{\boxed{=}} \frac{x^3-2x^2+5x+6}{x+2}$



➡ when remainder is zero "0", divisor is a factor of dividend.

(dividend) $f(x)$ = (quotient × divisor) + remainder

$$\begin{array}{r|l} & x^2-4x+13 \\ \hline x+2 & x^3-2x^2+5x+6 \\ & \underline{-x^3 \pm 2x^2} \\ & -4x^2+5x+6 \\ & \underline{\mp 4x^2 \mp 8x} \\ & 13x+6 \\ & \underline{-13x \pm 26} \\ & -20 \end{array}$$

remainder sign

$(x^2-4x+13) - \frac{20}{x+2}$

* We always have remainder $\boxed{\neq 0}$ when greatest power of "x" in dividend (nominator) is more than equal to greatest power of "x" in divisor (denominator).

Note : In order to reduce error : In dividend Make sure to all powers of x in sequence, From greatest to lowest, Even if any power is not present in between. eg. $2x^3+2x+6$ X

$2x^3+0x^2+2x+6$ ✓

Remember :

- When a polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$.
- When a polynomial $f(x)$ is divided by $(ax-b)$, the remainder is $f\left(\frac{b}{a}\right)$.
- For any polynomial $f(x)$, if $f(a)=0$ then the remainder when $f(x)$ is divided by $(x-a)$ is zero.

  Thus $(x-a)$ is a factor of $f(x)$.
- For any polynomial $f(x)$, if $f\left(\frac{b}{a}\right)=0$, then $(ax-b)$ is a factor of $f(x)$.

Quadratic Formula :

$$x = \frac{-b \pm \sqrt{\boxed{b^2-4ac}}}{2a}$$

if sum is negative, no real roots.

substitution variables with sign $(+,-)$.

Note :

$$P(x) > 0$$

$$(x-2)(2x^2+x+2) > 0$$

Let, $2x^2+x+2$

$a=2$

$b=1$

$c=2$

No real roots

hence, $x-2>0 \Rightarrow \boxed{x>2}$

Justification :

$$b^2-4ac = (1)^2-4(2)(2)$$
$$= 1-16$$
$$= -15.$$

Binomial Expansion :-

$$(a+b)^n = {}^nC_0 (a)^{n-0}(b)^0 + {}^nC_1 (a)^{n-1}(b)^1 + {}^nC_2 (a)^{n-2}(b)^2 + \dots\dots$$

Replace "a" by 1

$$(1+b)^n = {}^nC_0\, b^0 + {}^nC_1\, b^1 + {}^nC_2\, b^2 + \dots$$

$\boxed{\begin{matrix} {}^{-2}C_1 = \text{error} \\ {}^{\frac{1}{2}}C_1 = \text{error} \end{matrix}}$

$\boxed{\begin{matrix} {}^nC_0 = 1 \\ {}^nC_1 = n \end{matrix}}$ $\quad \boxed{{}^nC_2 = \frac{n(n-1)}{2!}}$ $\quad \boxed{{}^nC_3 = \frac{n(n-1)(n-2)}{3!}}$

$$\boxed{{}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{4!}}$$

must be $\boxed{+1}$

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2!}b^2 + \frac{n(n-1)(n-2)}{3!}b^3 + \dots$$

use [* n must be fraction

* n must be -ve value] $\Rightarrow (1+x)^{-2}$ or $(1+x)^{\frac{3}{2}}$

$3! = 3\times2\times1$

$4! = 4\times3\times2\times1$

$5! = 5\times4\times3\times2\times1$

$\boxed{\begin{matrix} b+ax \\ b\left(1+\frac{a}{b}x\right) \end{matrix}}$ $\quad \boxed{\begin{matrix} \frac{1}{b}+ax \\ \frac{1}{b}\left(1+bax\right) \end{matrix}}$ $\quad \boxed{\begin{matrix} (a\times b)^n \\ = a^n \times b^n \end{matrix}}$

Partial Fraction :-

① Linear Non-repeat $\boxed{\frac{x}{(x+1)(x+2)}} \to$

② Linear repeat $\boxed{\frac{x+1}{(x+1)(x+2)^2}} \longrightarrow$ $\boxed{\text{Focus on denominator}}$

③ Quadratic Non-repeat $\boxed{\frac{2x+3}{(x-1)(x^2+2)}} \to$

④ Quadratic repeat $\boxed{\frac{3x+1}{(x-1)(x^2+2)^2}}$

$\hookrightarrow$ not in A2 9709 $\leftarrow$

$$\frac{\frac{A}{B}}{C} = \frac{A}{BC}$$

$$\frac{A}{\frac{B}{C}} = \frac{AC}{B}$$

① Linear Non-repeat

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

② Linear repeat

$$\frac{x+1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

③ Quadratic Non-repeat

$$\frac{2x+3}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+2)}$$

examples of Improper fraction: $\frac{x^2+1}{x^2-2x+1}$, $\frac{(x+1)(x-1)(x+2)}{(x+5)(x+6)}$, $\frac{x^3+2x+1}{x^2+1}$

$1 + x^2 = 0$

$x^2 = -1$

$x = \sqrt{-1}$

error

For ①,②,③:-

When greatest power of x in nominator is equal or greater than greatest power of x in denominator, it is said to be Improper fraction.

In order to convert Improper fraction → partial fraction

Do long division....

$$\frac{x^2+3x+3}{(x+1)(x+3)}$$

$$= \frac{x^2+3x+3}{x^2+4x+3}$$

$$\begin{array}{r|l} & 1 \\ x^2+4x+3 & x^2+3x+3 \\ & \underline{-\,x^2 \pm 4x \pm 3} \\ & -x \end{array}$$

$$1 + \frac{-x}{x^2+4x+3}$$

$$1 + \frac{-x}{(x+1)(x+3)}$$

⇒ Now, do partial fraction to this fraction.

Improper fraction → long division → partial fraction ← Bionomial expansion

↳ expand till x^2

Q ⇒

Example:

$$\frac{2x^4}{(x-3)(x+2)^2}$$ ← improper fraction $x^4 \gg x^3$

$$\frac{2x^4}{(x-3)(x^2+4x+4)} \Rightarrow \frac{2x^4}{x^3+4x^2+4x-3x^2-12x-12} \Rightarrow \frac{2x^4}{x^3+x^2-8x-12}$$

$$\begin{array}{r|l} & 2x-2 \\ \hline x^3+x^2-8x-12 & 2x^4+0x^3+0x^2+0x+0 \\ & \underset{-}{2x^4}+\underset{-}{2x^3}\underset{+}{-16x^2}\underset{+}{-24x} \\ \hline & -2x^3+16x^2+24x+0 \\ & \underset{+}{-2x^3}\underset{+}{-2x^2}\underset{-}{+16x}\underset{-}{+24} \\ \hline & 18x^2+8x-24 \end{array}$$

hence $$(2x-2)+\frac{18x^2+8x-24}{x^3+x^2-8x-12}$$ → proper form.

$$(2x-2)+\boxed{\frac{18x^2+8x-24}{(x-3)(x+2)^2}} \quad -①$$

let

$$\frac{18x^2+8x-24}{(x-3)(x+2)^2}=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

multiply both sides with $(x-3)(x+2)^2$

$$18x^2+8x-24 = A(x+2)^2+B(x+2)(x-3)+C(x-3).$$

put $x=3$

$162 = 25A+0+0 \Rightarrow \boxed{A=\frac{162}{25}}$

put $x=-2$

$32 = 0+0+C(-5) \Rightarrow \boxed{C=-\frac{32}{5}}$

put $x=0$

$-24=4A-6B-3C \Rightarrow \boxed{B=\frac{288}{25}}$

put in ① equation.

$$\frac{2x^4}{(x-3)(x+2)^2} = (2x-2) + \frac{162}{25(x-3)} + \frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

$$\Rightarrow (2x-2) + \frac{162}{-75\left(1-\frac{x}{3}\right)} + \frac{288}{50\left(1+\frac{x}{2}\right)} - \frac{32}{20\left(1+\frac{x}{2}\right)^2}$$

$$= (2x-2) - \frac{54}{25}\left(1-\frac{x}{3}\right)^{-1} + \frac{144}{25}\left(1+\frac{x}{2}\right)^{-1} - \frac{8}{5}\left(1+\frac{x}{2}\right)^{-2}$$

— eq ②

Now expand

$$-\frac{54}{25}\left(1-\frac{x}{3}\right)^{-1} \Rightarrow -\frac{54}{25}\left[1+\frac{x}{3}+\frac{x^2}{9}\right] \Rightarrow -\frac{54}{25} - \frac{54}{75}x - \frac{6}{25}x^2$$

$$\frac{144}{25}\left(1+\frac{x}{2}\right)^{-1} \Rightarrow \frac{144}{25}\left[1-\frac{x}{2}+\frac{x^2}{4}\right] \Rightarrow \frac{144}{25} - \frac{72}{25}x + \frac{36}{25}x^2$$

$$-\frac{8}{5}\left(1+\frac{x}{2}\right)^{-2} \Rightarrow -\frac{8}{5}\left[1-x+\frac{3}{4}x^2\right] \Rightarrow -\frac{8}{5} + \frac{8}{5}x - \frac{6}{5}x^2$$

Now put in eq ②.

$$\frac{2x^4}{(x-3)(x+2)^2} = (2x-2) + 2 - 2x + 0x^2$$

$$= 2x - 2 - 2x + 0x^2 + 2$$

$$= 0.$$

Find Factors (Complete factorization) / Roots :

$$2x^3 - 15x^2 + 13x + 60 = 0$$ ← Max power of "x" greater than 2.

Let

$f(x) = 2x^3 - 15x^2 + 13x + 60$

By trial and error method.

Solve :

Try put $x = 1$.

$p(1) = 2(1)^3 - 15(1)^2 + 13(1) + 60 = \boxed{60}$. getting closer to zero.

put $x = 2$

$p(2) = 2(2)^3 - 15(2)^2 + 13(2) + 60 = \boxed{42}$ "sign of motivation"

put $x = 3$.

$p(3) = 2(3)^3 - 15(3)^2 + 13(3) + 60 = \boxed{18}$

Cool

put $x = 4$

$p(4) = 2(4)^3 - 15(4)^2 + 13(4) + 60 = \boxed{0}$ hence 4 is a root. ① $(x-4)$ is factor.

* If $f(x)$ value increases then go reverse. $1 \to 0 \to -1 \to -2$ instead of $0 \to 1 \to 2 \to 3$ etc.

As $(x-4)$ is a factor ⟶ remainder must be zero.

$$
\begin{array}{r|l}
 & 2x^2 - 7x - 15 \\
x-4 & 2x^3 - 15x^2 + 13x + 60 \\
 & \cancel{2x^3} \underset{+}{-} 8x^2 \\
\hline
 & -7x^2 + 13x + 60 \\
 & \cancel{-7x^2} \underset{-}{+} 28x \\
\hline
 & -15x + 60 \\
 & \cancel{-15x} \pm 60 \\
\hline
 & \text{zero.}
\end{array}
$$

$\Rightarrow 2x^2 - 7x - 15$

$= 2x^2 - 10x + 3x - 15$

$= 2x(x-5) + 3(x-5)$

$=$ ② $(2x+3)(x-5)$ ③

root $= -\frac{3}{2}$ root $= 5$.

hence

roots $= -\frac{3}{2}, 4, 5$.

"Finding Quadratic Equation from roots" :-

$$x^2 - \begin{pmatrix} \text{Sum of any} \\ \text{2 roots} \end{pmatrix} x + \begin{pmatrix} \text{Product of} \\ \text{those 2} \\ \text{roots} \end{pmatrix} = 0$$

Example

$x^2 + 5x + 4 = 0$

$\Rightarrow x^2 + 4x + 1x + 4 = 0$

$\Rightarrow x(x+4) + 1(x+4) = 0$

$\Rightarrow (x+1)(x+4) = 0$

$\Rightarrow x = -4 \; , \; x = -1 .$

Sum of roots $= (-4) + (-1) = -5 .$

Product of roots $= (-4) \times (-1) = 4 .$

$x^2 - (-5)x + (4) = 0$

$x^2 + 5x + 4 = 0 .$

Find cubic equation if 3 roots are 3, 2, 1

Sol lets $\boxed{\text{take } 3 \;\&\; 2}$

Sum of roots $= 3+2 = 5$

Product of roots $= 3\times2 = 6$

So quadratic equation is $x^2 - 5x + 6 = 0$

Now use root ① → factor → $(x-1)$.

$$(x-1)(x^2-5x+6) = 0$$

$$\Rightarrow x^3 - 5x^2 + 6x - x^2 + 5x - 6 = 0$$

$$\Rightarrow x^3 - 6x^2 + 11x - 6 = 0.$$

Logarithms and exponential functions :-

① $\log_{10} x = \log x$

$\log_e x = \ln x$

$\log_{10} 5$ → Index. → base

② $\log_{10} 10 = 1 \Rightarrow \log 10 = 1$

$\log_e e = 1 \Rightarrow \ln e = 1.$

$\log_3 3 = 1 \Rightarrow \log_a a = 1.$

③ $\log(x \cdot y) = \log(x) + \log(y)$.

④ $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

→ There is no such rule for addition & subtraction for logarithms.

⇒ we take log to bring power down.

⑤ $\log x^y = y \times \log x$.

⑥ $\log 1 = 0$

$\ln 1 = 0$

[redacted]

(11) $\log x^{-y} = \log\left(\frac{1}{x^y}\right)$

or

$-y \log x$

or

$y \log \frac{1}{x}$

⑦ $\log 0 = \infty$

$\ln 0 = \infty$

⑧ $\log(0 < x < 1) = -ve$ → hence

$x \cdot \log(0.5) < 1$

$x > \frac{1}{\log(0.5)}$

basic rule:

direction changes when negative value move to other (multiply or divide).

In short, when negative is (multiply or divide) on both sides.

⑨ $\log(x > 1) = +ve$

⑩ $\log(x \leqslant 0) = \infty$

→ The logarithmic function is the inverse of the exponential function to the same base.

→ Must "Read, Read, Read" "9702 Osama log pdf" as well.

Common mistake:

↳ $3^{x+2} = 3^x \pm 3^2$ → Due to ⊕ or ⊖ sign, we can't take log.

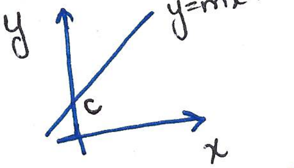
$\log(3^{x+2}) = \log(3^x) \pm \log(3^2)$ → wrong.

$y = mx + c$

coordinates

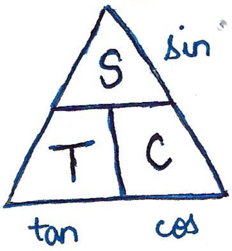
→ intercept at y axis

↳ gradient



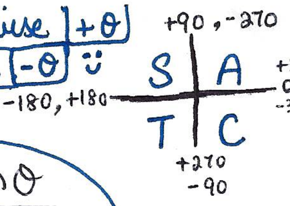
→ important for log chapter.

Trigonometry :-



anticlockwise | $+\theta$

clockwise | $-\theta$



$\sin\theta = \tan\theta\cos\theta$ ∵ $\cos\theta = \dfrac{\sin\theta}{\tan\theta}$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\mathrm{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\tan\theta\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\tan\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta, \cos^2\theta$
$\sin\theta, \cos\theta$ → $+1$ highest, -1 lowest

As

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

$\div \cos^2\theta$ ← ①

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

$\div \sin^2\theta$ ← ②

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta = \mathrm{cosec}^2\theta}$$

Fundamental Laws :-

① $\sin(A+B) = \sin A\cos B + \cos A\sin B$ – ① eq

② $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

③ $\cos(A+B) = \cos A\cos B - \sin A\sin B$ – ② eq

④ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\cdot\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\cdot\tan B}$$

eq ①

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

Replace "B" by "A"

$\sin(A+A) = \sin A\cos A + \cos A\sin A$

$$\boxed{\sin 2A = 2\sin A\cdot\cos A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

eq ③

$\cos 2A = \cos^2 A - \sin^2 A$

➡ Replace "$\sin^2 A = 1 - \cos^2 A$"

$\cos 2A = \cos^2 A - [1 - \cos^2 A]$

$$\boxed{\cos 2A = 2\cos^2 A - 1}$$

eq ②

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

Replace "B" by "A"

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$ → eq ③

eq ③

$\cos 2A = \cos^2 A - \sin^2 A$

➡ Replace "$\cos^2 A = 1 - \sin^2 A$"

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

acute $< 90°$

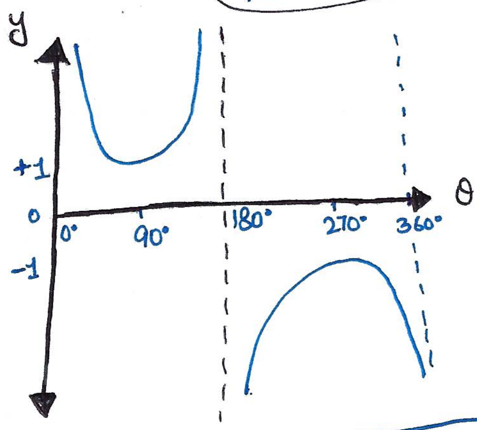
$90° < \theta < 180°$ obtuse

$180° < \theta < 360°$ reflex

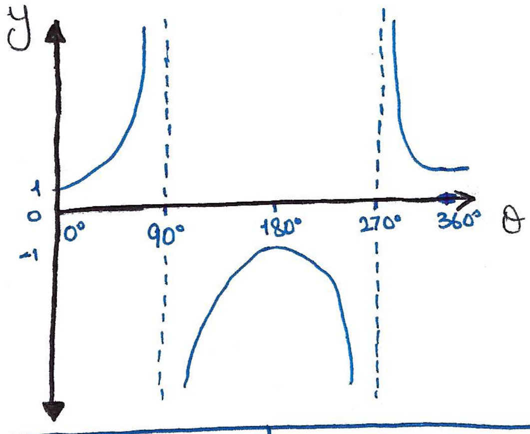
$2\pi = 360°$

Cosecant, secant and cotangent: Graphs

$\text{Cosec}\,\theta = \frac{1}{\sin\theta}$



$\sec\theta = \frac{1}{\cos\theta}$

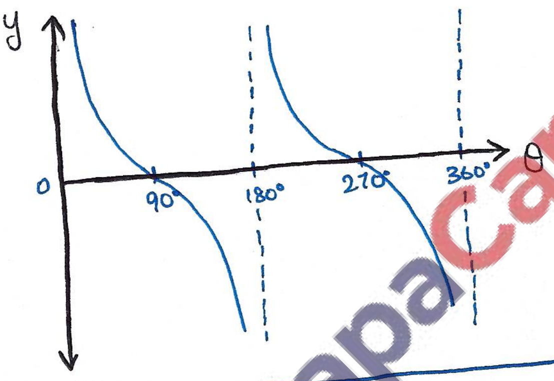


Note :- $N = 105e^{\pm \sin 0.02t}$

For max value, take max value of sin(xyz) which is $\boxed{+1}$

For min value, take min value of sin(xyz) which is $\boxed{-1}$

$\cot\theta = \frac{1}{\tan\theta}$



Note

$0° \leq \theta \leq 360°$

$\cos(\theta - 73.7°) = \frac{15}{25}$

$\alpha = \cos^{-1}\left(\frac{15}{25}\right) = 53.1°$

So $\theta - 73.7° = 53.1,\ ,\ -53.1$

$\theta = \boxed{126.8°},\ \boxed{20.6°}$

$\boxed{-73.7° \leq \theta - 73.7° \leq 286.3°}$

Expressing $a\sin\theta + b\cos\theta$ in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$:

- $a\sin\theta + b\sin\theta \equiv R\sin(\theta + \alpha)$ where and $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$
- $a\sin\theta + b\cos\theta \equiv R\cos(\theta - \alpha)$ where and $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$
- $a\sin\theta - b\cos\theta \equiv R\sin(\theta - \alpha)$ where and $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$
- $a\cos\theta - b\sin\theta \equiv R\cos(\theta + \alpha)$ where and $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

- $\sin(90° - \theta) \equiv \cos\theta$
- $\tan(90° - \theta) \equiv \cot\theta$
- $\sec(90° - \theta) = \text{cosec}\,\theta$
- $\cos(90° - \theta) \equiv \sin\theta$
- $\cot(90° - \theta) \equiv \tan\theta$
- $\text{cosec}(90° - \theta) \equiv \sec\theta$.

Differentiation :-

① $y = x^n$

$\frac{dy}{dx} = nx^{n-1}.$

② $y = (2x^2+5)^{10}$

$\frac{dy}{dx} = 9(2x^2+5)^9 \times (4x+0)$

$= 40x\,(2x^2+5)^9.$

③ $y = e^{5x}$

$\frac{dy}{dx} = e^{5x} \times \frac{d}{dx}(5x)$

$= 5e^{5x}$
↳ exponential

④ $y = \sin(10x)$

$\frac{dy}{dx} = \cos(10x) \cdot \frac{d}{dx}(10x)$

$= 10\cos(10x).$

⑤ $y = \cos(20x^2)$

$\frac{dy}{dx} = -\sin(20x^2) \cdot \frac{d}{dx}(20x^2)$

$= -40x\,\sin(20x^2).$

⑥ $y = \tan(3x)$

$\frac{dy}{dx} = \sec^2(3x) \times \frac{d}{dx}(3x)$

$= 3\sec^2(3x).$

⑦ $y = \ln(x^2+3)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2+3)}{x^2+3}$

$\frac{dy}{dx} = \frac{2x}{x^2+3}.$

⑧ $y = \sin^3(4x^2+1)$

$\boxed{\text{Power} \rightarrow \text{base} \rightarrow \text{angle.}}$

$\frac{dy}{dx} = 3\sin^2(4x^2+1) \cdot \cos(4x^2+1) \cdot \frac{d}{dx}(4x^2+1).$

$= 3\sin^2(4x^2+1)\cos(4x^2+1) \cdot 8x$

$= 24x\sin^2(4x^2+1)\cos(4x^2+1).$

Exp $y = \sqrt{\cos(20x)}$

$y = [\cos(20x)]^{\frac{1}{2}}.$

$y = \cos^{\frac{1}{2}}(20x)$

$\frac{dy}{dx} = \frac{1}{2}\cos^{-\frac{1}{2}}(20x) \cdot -\sin(20x) \cdot 20.$

$= \frac{-10\sin(20x)}{\sqrt{\cos(20x)}}$

For max  $-x^2$

- $\frac{dy}{dx}$ = just before is +ve and just after is -ve.
- $\frac{d^2y}{dx^2} = -ve$

AS Maths

For min  $+x^2$.

- both points are opposite

Quotient Rule :

order matters

$$\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\cdot\frac{d}{dx}(U) - U\cdot\frac{d}{dx}(V)}{V^2}$$

eg. $y = \dfrac{\boxed{3x^2} \rightarrow ⓤ}{\boxed{2x-1} \rightarrow ⓥ}$ → must

$$\frac{dy}{dx} = \frac{(2x-1)(6x) - 3x^2(2)}{(2x-1)^2}$$

$$\frac{dy}{dx} = \frac{6x^2-6x}{(2x-1)^2}$$

$$\frac{dy}{dx} = \frac{6x(x-1)}{(2x-1)^2}$$ →

differentiation must be in one single simplied fraction.

Product Rule :

order doesn't matter

$$\frac{d}{dx}(U.V) = U.\frac{d}{dx}(V) + V\frac{d}{dx}(U)$$

any one can be $\boxed{U/V}$

eg. $y = \boxed{x^2}\boxed{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = x^2\left[\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(2x)\right] + \sqrt{x^2+1}\,(2x)$$

$$= x^3(x^2+1)^{-\frac{1}{2}} + 2x\sqrt{x^2+1}$$

$$= \frac{x^3}{\sqrt{x^2+1}} + 2x\sqrt{x^2+1}$$

$$= \frac{x^3 + 2x(x^2+1)}{\sqrt{x^2+1}} = \frac{3x^3+2x}{\sqrt{x^2+1}} = \frac{x(3x^2+2}{\sqrt{x^2+1}}$$

Integration :-

- $\int e^{3x}\,dx$

$= \frac{e^{3x}}{3} + c$ → always when limits not present. indefinite integration

- $\int (e^{2x+1})^2\,dx$

$= \int e^{4x+2}\,dx$

$= \frac{e^{4x+2}}{4} + c$

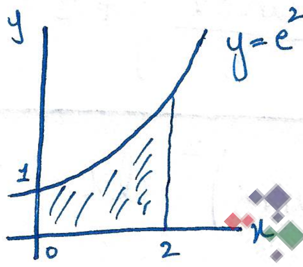
- $\int \frac{e^{2x+3}}{e^{5x}}\,dx$

$= \int e^{2x+3-5x}\,dx$

$= \int e^{3-3x}\,dx$

$= \frac{e^{3-3x}}{-3} + c$

- Area



across y-axis

$\int_{y_1}^{y_2} x\,dy$

Shaded area $= \int_{x_1}^{x^2} y\,dx$

$= \int_0^2 e^{2x}\,dx$

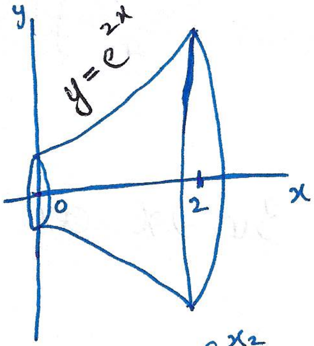
$= \left[\frac{e^{2x}}{2}\right]_0^2 = \frac{1}{2}[e^4 - e^0]$

Area $= \frac{1}{2}(e^4 - 1)$ unit^2

- Volume

across y-axis

$\pi\int_{y_1}^{y_2} x^2\,dy$



Volume $= \pi\int_{x_1}^{x_2} y^2\,dx$

$= \pi\int_0^2 (e^{2x})^2\,dx$

$= \pi\int_0^2 (e^{4x})\,dx$

$= \pi\left[\frac{e^{4x}}{4}\right]_0^2$

Volume $= \frac{\pi}{4}[e^8 - 1]$ unit^3

- $$\int \frac{f'(x)}{f^1(x)}\,dx = \ln\left[f(x)\right] + c$$

Note
① $f(x)$ is denominator
② Derivative of $f(x)$ in Numerator
③ $f(x)$ power is "1"

$$\int \frac{2}{2x+1}\,dx = \ln(2x+1) + c$$

- $$\int \sin^2 2x\,dx$$

half

$$\left[\begin{array}{l} \cos 2x = 1 - 2\sin^2 x \\ \cos 4x = 1 - 2\sin^2(2x) \\ \sin^2 2x = \dfrac{1-\cos 4x}{2} \end{array}\right]$$

$$\int \frac{1-\cos 4x}{2}\,dx$$

$$\frac{1}{2}\int 1 - \cos 4x\,dx$$

$$\frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + c$$

- $$\int \cos^2 4x\,dx$$

half

$$\left[\begin{array}{l} \cos 2x = 2\cos^2 x - 1 \\ \cos 8x = 2\cos^2 4x - 1 \\ \dfrac{1+\cos 8x}{2} = \cos^2 4x \end{array}\right]$$

$$\int \frac{1+\cos 8x}{2}\,dx$$

$$\frac{1}{2}\int 1 + \cos 8x\,dx$$

$$\frac{1}{2}\left[x + \frac{\sin 8x}{8}\right] + c$$

- $$\int \sin 2x\,dx = -\frac{\cos x}{2} + c$$

- $$\int \cos 3x\,dx = \frac{\sin 3x}{3} + c$$

- $$\int \sec^2 4x\,dx = \frac{\tan 4x}{4} + c$$

- $$\int \sin 4x\,dx = -\frac{\cos 4x}{4} + c$$

$$\int (1+\tan x)^2\,dx$$

$$\int (1)^2 + 2(1)(\tan x) + (\tan x)^2\,dx$$

$$\int 1 + 2\tan x + \tan^2 x\,dx$$

$$\int 1\,dx + \int 2\tan x\,dx + \int \tan^2 x\,dx$$

$$\int 1\,dx + 2\int \tan x\,dx + \int \sec^2 x - 1\,dx$$

$$\int 1\,dx - 2\int \frac{-\sin x}{\cos x}\,dx + \int \sec^2 x - 1\,dx$$

$$\cancel{x} - 2\ln\cos x + \tan x - \cancel{x} + c$$

Implicit function :

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^2) = 2x$
- $\frac{d}{dx}(x^3) = 3x^2$
- $\frac{d}{dx}(e^{3y}) = e^{3y} . \ 3\frac{dy}{dx}$

- $\frac{d}{dx}(y) = \frac{dy}{dx}$
- $\frac{d}{dx}(y^2) = 2y^{1}\frac{dy}{dx}$.
- $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$.

Ex $x^3 + xy^2 = 5x$

$3x^2 + x\left(2y\frac{dy}{dx}\right) + y^2(1) = 5(1)$.

$3x^2 + 2xy\frac{dy}{dx} + y^2 = 5$.

$$\frac{dy}{dx} = \frac{5 - y^2 - 3x^2}{2xy} .$$

Parametric equations :

$0 < t < \frac{\pi}{2}$.

$x = 5\cos 3t$

$y = 2\sin 3t$

ⓐ Find $\frac{dy}{dx}$ in terms of t

$\frac{dx}{dt} = 5\cdot(-\sin 3t)\cdot 3 = -15\sin 3t$

$\frac{dy}{dt} = 2(\cos 3t)\times 3 = 6\cos 3t$.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} .$$

$$\frac{dy}{dx} = \frac{6\cos 3t}{-15\sin 3t} = \frac{2\cos 3t}{-5\sin 3t}$$

ⓑ Find coordinates of the stationary points.

$\downarrow$

$\frac{dy}{dx} = 0$.

$$0 = \frac{2\cos 3t}{-5\sin 3t}$$

$$0 = \cos 3t$$

$$\cos^{-1}(0) = 3t$$

$$\frac{1}{2}\pi = 3t$$

$$\frac{1}{6}\pi = t .$$

$$\longrightarrow$$

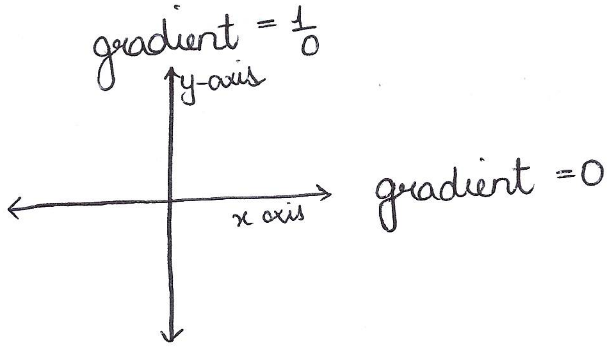
$$x = 5\cos 3(\tfrac{1}{6}\pi)$$

$$\boxed{x = 0}$$

$$y = 2\sin(\tfrac{1}{2}\pi)$$

$$\boxed{y = 2}$$

hences, coordinate $(0, 2)$.



The parametric equations of a curve are

$x = e^{-t}\cos t$ $\qquad$ $y = e^{-t}\sin t$

Show that $\frac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

do not show examiner at start.

$$\tan\left(t - \tfrac{1}{4}\pi\right) = \frac{\tan t - \tan\frac{1}{4}\pi}{1 + \tan t \tan\frac{1}{4}\pi} = \frac{\tan t - 1}{1 + \tan t}.$$

go reverse to get some extra idea

Sol

$$\frac{dx}{dt} = e^{-t}\left[-\sin t\right] + \cos t\left[e^{-t}.-1\right] = e^{-t}\left[-\sin t - \cos t\right]$$

$$\frac{dy}{dx} = e^{-t}\left[\cos t\right] + \sin t\left[e^{-t}.-1\right] = e^{-t}\left[\cos t - \sin t\right].$$

By chain rule ← must & always.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos t - \sin t}{-\sin t - \cos t} = \frac{-\left[-\cos t + \sin t\right]}{-\left[\sin t + \cos t\right]}$$

$$\frac{dy}{dx} = \frac{\sin t - \cos t}{\cos t + \sin t} \quad \div \text{ by } \cos t.$$

$$= \frac{\frac{\sin t}{\cos t} - \frac{\cos t}{\cos t}}{\frac{\cos t}{\cos t} + \frac{\sin t}{\cos t}} = \frac{\tan t - 1}{1 + \tan t}$$

now write those "reverse" steps.

$$= \frac{\tan t - \tan\frac{1}{4}\pi}{1 + \tan t \tan\frac{1}{4}\pi}$$

$$\frac{dy}{dx} = \tan\left(t - \tfrac{1}{4}\pi\right) \quad //$$

proved

Exact Values of Trigonometric Functions

Angle θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
Degrees	Radians			
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	Not defined.

Integrating more trigonometric functions :

Standard trigonometrical relationships, such as identities and double-angle formulae, can be used to rearrange functions so that they are integrable.

Integration of rational functions :

To integrate difficult algebric fractions, first look to see if you can split the fraction into partial fractions, where each partial fraction is integrable.

- $\displaystyle\int_3^4 \frac{5x+5}{(x-2)(x+3)}\,dx$

let $\dfrac{5x+5}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$

Multiply $(x-2)(x+3)$

$5x+5 = A(x+3) + B(x-2)$

put $x=2$

$10+5 = 5A + 0$

$\boxed{A=3}$

put $x=-3$

$-15+5 = A(0) + B(-5)$

$-10 = -5B$

$\boxed{B=2}$

$$\frac{5x+5}{(x-2)(x+3)} = \frac{3}{x-2} + \frac{2}{x+3}$$

$$\int_3^4 \frac{5x+5}{(x-2)(x+3)}\,dx \longrightarrow \int_3^4 \frac{3}{x-2} + \frac{2}{x+3}\,dx.$$

$$= 3\int_3^4 \frac{1}{x-2}\,dx + 2\int_3^4 \frac{1}{x+3}\,dx.$$

$$= 3\Big[\ln(x-2)\Big]_3^4 + 2\Big[\ln(x+3)\Big]_3^4$$

$$= 3\ln 2 + 2\ln\frac{7}{6}.$$

$$= \ln 8 + \ln\frac{49}{36} = \ln\left(\frac{98}{9}\right)$$

specimen 2020 Q10 last part ↓

$\displaystyle\int_3^2 5x\,dx.$

lower limit greater than upper limit

So, $\displaystyle\int_2^3 -5x\,dx$

Now solve just like normal integration.

x approaches ....

very large or very small → ∞

∞ $\times$ $+$ $-$ $\div$ give ∞

$e^{\pm x} \neq 0$

$\frac{1}{\infty} = 0$

$\frac{1}{0} = \infty$

$e^{\infty} = \infty$

$e^{-\infty} = 0$

Q $\left.\begin{array}{l} 2x = 3x - x \\ 4x = 3x + x \end{array}\right] \rightarrow$ use.

Show

$$\sin 3x \cos x = \frac{1}{2}(\sin 2x + \sin 4x)$$

As $\sin 2x = \sin(3x - x)$.

$\sin 2x = \sin 3x \cos x - \cos 3x \sin x$ - ①.

As $\sin 4x = \sin(3x + x)$.

$\sin 4x = \sin 3x \cos x + \cos 3x \sin x$ -②

Adding ① + ②

$\sin 2x + \sin 4x = 2 \sin 3x \cos x$.

$\boxed{\div \text{ by } 2}$ $\sin 3x \cos x = \frac{1}{2}(\sin 2x + \sin 4x)$

Integration by parts :

$$\int u.v \, dx = u \times \int v dx - \int \left(\int v dx \times \frac{d}{dx}(u) \right) dx.$$

U ↓ V

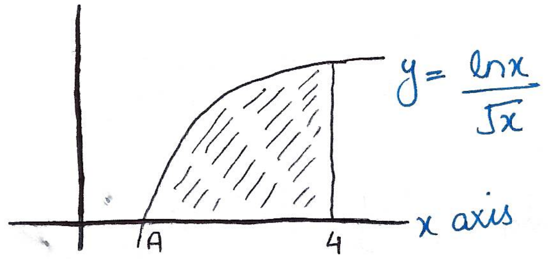
- L → logarithmic functions → $\ln(x)$, $\log_2(x)$ etc
- I → Inverse trig. functions → $\tan^{-1}(x)$, $\sin^{-1}(x)$ etc
- A → Algebraic functions → x, $3x^2$, $5x^2$ etc.
- T → Trig. functions → $\cos(x)$, $\tan(x)$ etc
- E → exponential functions → e^x, 2^x etc.

$\int_1^a \ln(2x) dx$

as we can not integrate ln so do like this for all similar cases.

$\int_1^a \underset{u}{\ln(2x)} \cdot \underset{v}{1} \, dx$ Now solve. ☺

Q



Find area shaded:

put $y=0$.

$0 = \frac{\ln x}{\sqrt{x}}$.

$0 = \ln x$ $\quad A(1,0)$

$\boxed{x=1}$

$$\text{Area} = \int_{x_1}^{x_2} y\,dx$$

$$\text{Area} = \int_1^4 \frac{\ln x}{\sqrt{x}}\,dx.$$

$$= \int_1^4 \ln x\,(x^{-\frac{1}{2}})\,dx.$$

$$= \ln x \cdot \int x^{-\frac{1}{2}}dx - \int\int x^{-\frac{1}{2}}dx \cdot \frac{d}{dx}(\ln x)\,dx$$

$$= \ln x \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \int \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{x}\,dx$$

$$= 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,dx$$

$$= \left[2\sqrt{x}\ln x - 2 \cdot \frac{x^{+\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$\text{Area} = [4\ln 4 - 8] - [0-4]$$

$$= 4\ln 4 - 8 + 4.$$

$$= 4\ln 2^2 - 4$$

$$= (8\ln 2 - 4)\ \text{unit}^2.$$

Note:

$(\sqrt{2})^5 = 2^{\frac{5}{2}} = 2^{2.5}$

$2^2 \times 2^{0.5}$

$2^2 \times \sqrt{2}$

$4 \times \sqrt{2}$.

upper limit must be greater than lower limit. if not, multiply the equation with negative to switch limits

↳ IMP.

$(\sqrt{2})^1 = \sqrt{2}$

$(\sqrt{2})^2 = 2$

$(\sqrt{2})^3 = 2\sqrt{2}$

$(\sqrt{2})^4 = 4$

$\int_{\text{lower limit}}^{\text{upper limit}}$

Q Let $I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(4-u)}\,du$

(ii) Hence show that $I = \frac{1}{2}\ln 3$.

(i) $x = u^2$ $\quad dx = 2u\,du$.

$\frac{dx}{du} = 2u$

limits:

$u = \sqrt{x}$
$u = \sqrt{4}$
$u = 2$

$u = \sqrt{x}$
$u = \sqrt{1}$
$u = 1$.

$$I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\, dx.$$

$$I = \int_1^2 \frac{1 \times 2u\,du}{u^2(4-u)}$$

$$= \int_1^2 \frac{2}{u(4-u)}\, du$$

↑
no "x" at all → must.

(ii) $$\frac{2}{u(4-u)} = \frac{A}{u} + \frac{B}{4-u}$$

Multiply by $u(4-u)$..

$$2 = A(4-u) + Bu.$$

put $u = 4$

$$2 = 0 + 4B$$

$$B = \frac{1}{2}.$$

put $u = 0$

$$2 = 4A + 0$$

$$A = \frac{1}{2}.$$

$$\boxed{\frac{2}{u(4-u)} = \frac{1}{2u} + \frac{1}{2(4-u)}}$$

Now integrate both sides

$$\int_1^2 \frac{1}{2u}\, du + \frac{1}{2}\int_1^2 \frac{1}{4-u}\, du.$$

$$\frac{1}{2}\int_1^2 \frac{2}{2u}\, du + \frac{1}{2}\int_1^2 \frac{-1}{4-u}\, du.$$

$$\frac{1}{2}\Big[\ln(2u)\Big]_1^2 - \frac{1}{2}\Big[\ln(4-u)\Big]_1^2$$

$$\frac{1}{2}\Big[\ln(4) - \ln(2)\Big] - \frac{1}{2}\Big[\ln 2 - \ln 3\Big]$$

$$\frac{1}{2}\Big[\ln 2\Big] - \frac{1}{2}\Big[\ln \frac{2}{3}\Big].$$

$$\frac{1}{2}\Big[\ln 3\Big]$$

Differential Equations:

Q: Find an expression for y in terms x

$y = 1$ when $x = 0$

$$\frac{dy}{dx} = \frac{y^3+1}{y^2}$$

variable separation.

$$y^2\, dy = (y^3+1)\, dx$$

$$\int \frac{y^2}{y^3+1}\, dy = \int 1\, dx.$$

$$\frac{1}{3}\int \frac{3y^2}{y^3+1}\, dy = \int 1\, dx.$$

$$\frac{1}{3}\ln(y^3+1) + c = x.$$

$$\frac{1}{3}\ln(1^3+1) + c = 0$$

$$\frac{1}{3}\ln(2) + c = 0$$

$$c = -\frac{1}{3}\ln 2.$$

$$\frac{1}{3}\ln(y^3+1) + \left(-\frac{1}{3}\ln 2\right) = x.$$

$$\frac{1}{3}\ln(y^3+1) - \frac{1}{3}\ln 2 = x.$$

$$\frac{1}{3}\ln\left(\frac{y^3+1}{2}\right) = x.$$

$$\cancel{\ln}\left(\frac{y^3+1}{2}\right) = \cancel{\ln}\, e^{3x}$$

$$y^3+1 = 2e^{3x}$$

$$y = \sqrt[3]{2e^{3x}-1}$$

We can not integrate improper fraction. So, convert it by doing long division to get proper fraction

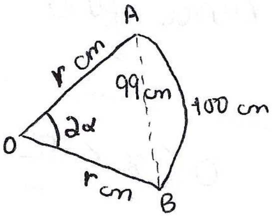
Integration using substitution :

- Before making a variable substitution, first write down everything in the integral in terms of the new variable, u, (including the "dx" term) and then rewrite the integral completely in one go.
- When dealing with definite integrals, make sure you express the limits in terms of the new variable, u.
- For indefinite integrals if your original integral was in terms of x, you need to express your answer also in terms of x.

Numerical solution of equations :

P-3 Oct 2002

① how α satisfies Equation $\frac{99}{100}x = \sin x$

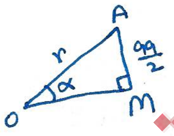


$S = r\theta$

$100 = r\,2\alpha$

$50 = r\alpha$

$r = \frac{50}{\alpha}$



$\frac{\text{opp}}{\text{hyp}} = \sin\theta$

$\frac{99 \div 2}{r} = \sin\alpha$

$\frac{99 \div 2}{50 \div \alpha} = \sin\alpha$

$\frac{99}{2} = \frac{50}{\alpha} \times \sin\alpha$

$\frac{99}{100}\alpha = \sin\alpha$

hence α satisfies equation $\boxed{\frac{99}{100}x = \sin x}$

(ii) root lies b/w 0.1 & 0.5

$f(x) = \frac{99}{100}x - \sin x$

$f(0.1) = \frac{99}{100}(0.1) - \sin(0.1)$

$\rightarrow = -8.33 \times 10^{-4}$

$f(0.5) = \frac{99}{100}(0.5) - \sin(0.5)$

$\rightarrow = +0.0155$

As sign changes so root lies b/w 0.1 & 0.5. proved.

(iii) Show that it converges iterative formula to a root of the eq in ①

$\boxed{x_{n+1} = 50\sin x_n - 48.5x_n}$

Remove subscripts → $x = 50\sin x - 48.5x$

$49.5x = 50\sin x$

$\frac{99}{100}x = \sin x$ → yes it converges to a root (solution) of equation in part ①

(iv) → get concept

(i) Iterative formula

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$$

initial value → $x_1 = 1$

- show the result of each iteration correct to 3 d.p.
- root correct to 2 d.p

put $n=1$.

$$x_2 = \frac{2}{3}\left(x_1 + \frac{1}{x_1^2}\right)$$

$$x_2 = \frac{2}{3}\left(1 + \frac{1}{1^2}\right) = \frac{4}{3} = \boxed{1.333} \rightarrow \boxed{1.33}$$

$$x_3 = \frac{2}{3}\left(1.333 + \frac{1}{1.333^2}\right) = \boxed{1.264} \rightarrow \boxed{1.26}$$

$$x_4 = \frac{2}{3}\left(1.264 + \frac{1}{1.264^2}\right) = \boxed{1.260} \rightarrow \boxed{1.26} \rightarrow \text{constant}$$

$$x_4 = \frac{2}{3}\left(1.260 + \frac{1}{(1.260)^2}\right) = \boxed{1.260} \rightarrow \boxed{1.26}$$

Root is $\underline{\underline{1.26}}$.

(ii) State an ~~expression~~ equation satisfied by α, hence find the exact value of α. 2d.p.

Remove subscripts

$$x = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

$$\frac{3}{2}x = x + \frac{1}{x^2}.$$

$$\frac{1}{2}x = \frac{1}{x^2}$$

$$\frac{1}{2}x^3 = 1$$

$$x^3 = 2$$

$$x = \sqrt[3]{2} = \underline{\underline{1.26}}$$

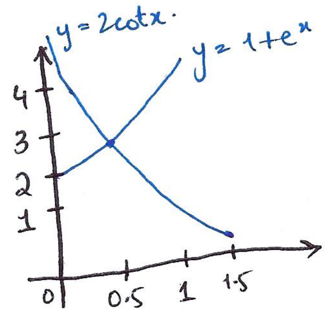
Q Num-Analysis $0 < x < \frac{\pi}{2}$

$$2\cot x = 1 + e^x$$

$y = 2\cot x$

$y = \frac{2}{\tan x}$

$y = 1 + e^x$.



It is clear that $2\cot x = 1 + e^x$ has one root between $0 < x < \frac{\pi}{2}$.

① Mode ↵

② 7 : table ↵

③ $f(x) = \frac{2}{\tan(x)}$ → Alpha)

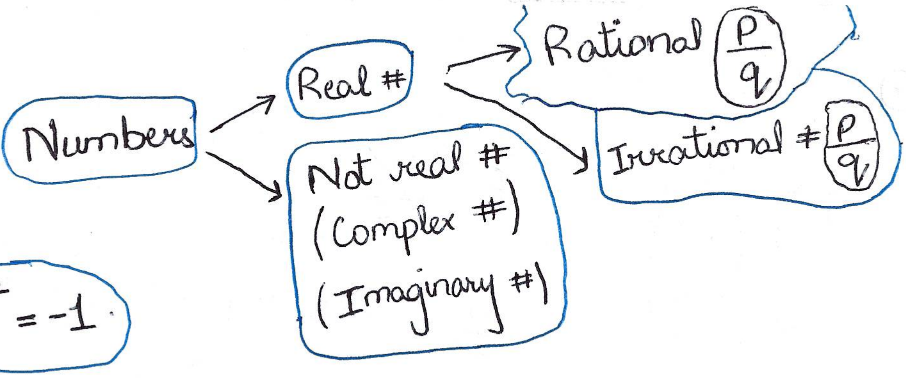
④ Start = 0

⑤ End = $\frac{\pi}{2}$

⑥ Step = 0.5

→ use calculator for graph questions

Complex :-



Complex #s

$\dot{\imath}$ i ← iota

$\boxed{i^2 = -1}$

$\boxed{Z, w, U}$

$Z = a + ib$

real part

complex part
imaginary part.

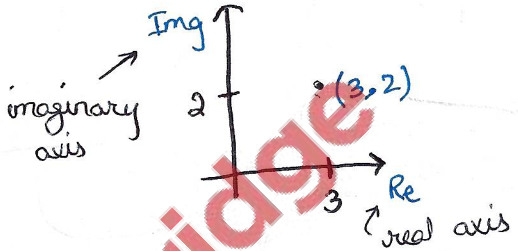
Conjugate

$Z = 5+3i \longrightarrow Z^* = 5-3i$

$U = -2-7i \longrightarrow U^* = -2+7i$.

Argand Diagram

$Z = 3+2i$ (or) $Z = 3 + i2$.



$\boxed{(a,b)}$ ↳ one complex #

examples

⊙ $(-2+3i)+(3+7i)$

$= 1+5i$.

⊙ $6(2-3i)-3(3+5i)$

$= 12-18i-9-15i$

$= 3-33i$

⊙ $(5+i)(2+i)$

$= 10+5i+2i+i^2$

$= 10+7i-1$

$= 9+7i$

⊙ $\frac{5+4i}{2+i}$

Rationalize

$= \frac{5+4i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{10-5i+8i-4i^2}{(2)^2-(i)^2}$

$= \frac{10+3i+4}{4-i^2} = \frac{10+3i+4}{4+1}$

$= \frac{14+3i}{5} = \frac{14}{5} + \frac{3}{5}i$

Complex
If one root is z then other root must be z*

⊙ $4z^2+2z+1=0$

a=4
b=2
c=1

also include signs.

$$Z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

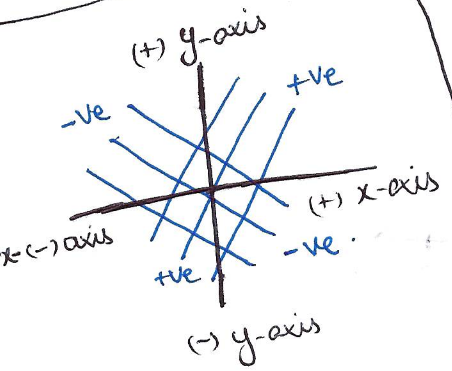
$$Z = \frac{-(2)\pm\sqrt{(2)^2-4(4)(1)}}{2(4)}$$

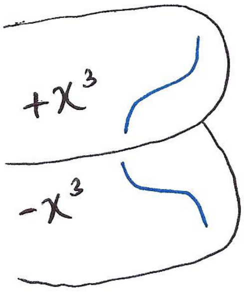
$$Z = \frac{-2\pm\sqrt{-12}}{8} = \frac{-2\pm\sqrt{12}\sqrt{-1}}{8}$$

$$= \frac{-2\pm\sqrt{12}\,i}{8} = \frac{-2\pm\sqrt{2^2\times3}\,i}{8}$$

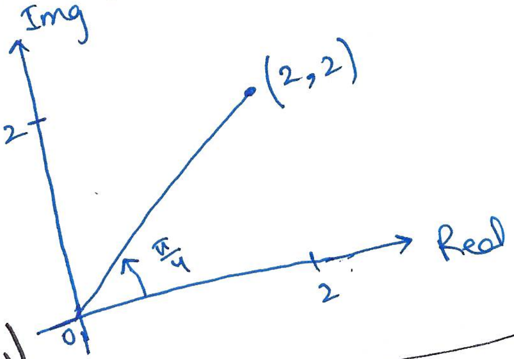
$$= \frac{-2\pm2\sqrt{3}i}{8} = \frac{2(-1\pm\sqrt{3}i)}{8}$$

$$= \frac{1}{4}(-1\pm\sqrt{3}i)$$





Argand diagram



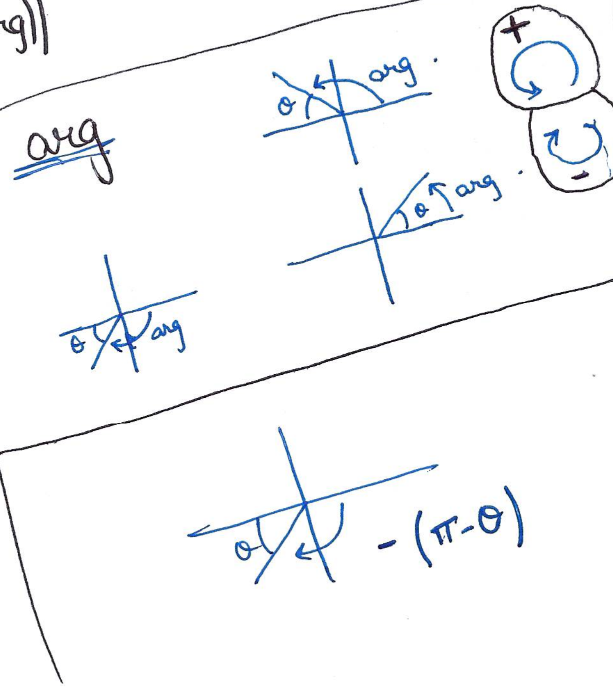
Q $Z = 2+2i$

- ⓐ $|Z|$ ← modulus ← start distance from origin
- ⓑ $arg(Z)$ ← argument ← shortest angle from +ve x-axis
- ⓒ Polar form ← $|Z|(\cos(arg)+i\sin(arg))$

ⓐ $|Z| = \sqrt{(2)^2+(2)^2}$
$= \sqrt{8} = 2\sqrt{2}$.

ⓑ $\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4} = arg(Z)$

ⓒ $2+2i = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$



Q $Z = 2 + 2i$

$W = -1 + \sqrt{3}\, i$

[illegible] [illegible] [illegible] [illegible] .....

ⓐ $Z^2 = ?$ ⓑ $Z^2 W^2 = ?$ ⓒ $\frac{W^2}{Z^3} = ?$

$|Z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\arg = \frac{\pi}{4}$

<u>Polar</u> → $Z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$|w| = \sqrt{1+3} = 2$.

$\theta = \frac{\pi}{3}$

$\arg = \frac{2\pi}{3}$.

$w = 2\left[\left(\cos\frac{2\pi}{3}\right) + i\sin\frac{2\pi}{3}\right]$.

ⓐ $Z^2 = (2\sqrt{2})^2\left[\cos\left(2\times\frac{\pi}{4}\right) + i\sin\left(2\times\frac{\pi}{4}\right)\right]$

$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

$= 8(0+i) = 8i$

ⓒ $\frac{W^2}{Z^3} = \frac{4\left(\cos\frac{4\pi}{3} + i\sin\frac{4}{3}\pi\right)}{\underset{\times\sqrt{2}}{16}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}$

$= \frac{\sqrt{2}}{8}\left(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right)$

ⓑ $Z^2 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$W^2 = 4\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right)$

multiply.

$Z^2 W^2 = 32\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)$

<u>Remember</u> :

modulus- argument form

eg. $(\underset{\uparrow\ \text{modulus}}{12}, \underset{\nwarrow\ \text{arg.}}{\tfrac{3}{4}\pi})$

Q $\sqrt{(3-4i)} = ?$

$\sqrt{(3-4i)} = \pm(a+ib)$

squaring both sides

$3-4i = (a+ib)^2$

$3-4i = a^2 + 2abi + (ib)^2$

$3-4i = a^2 + 2abi - b^2$

$3-4i = (a^2-b^2) + (2ab)i$

Compare

$a^2 - b^2 = 3 \Rightarrow$

$2ab = -4$

$ab = -2$

$b = -\frac{2}{a} \Rightarrow$

$a^2 - \left(\frac{-2}{a}\right)^2 = 3$

$a^2 - \frac{4}{a^2} = 3$

$a^4 - 4 = 3a^2$

$a^4 - 3a^2 = 4$

let $a^2 = x$

$x^2 - 3x = 4$

$x^2 - 3x - 4 = 0$

$x^2 - 4x + 1x - 4 = 0$

$x(x-4) + 1(x-4) = 0$

$(x+1)(x-4) = 0$.

$x = -1$, $x = 4$.

$\boxed{a^2 = -1}$, $a^2 = 4$

ignore $a = \pm 2$.

$b = -\frac{2}{a} \Rightarrow$

When $a = +2$

$b = -1$

when $a = -2$

$b = +1$.

$(+2-i)$

$(-2+i)$ <u>Ans.</u>

Q $u = 1 - \sqrt{3}i$

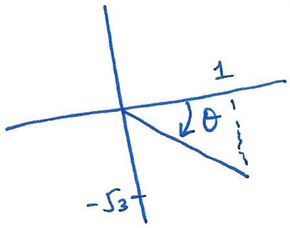
find

ⓐ $|u| = \sqrt{(1)^2 + (-\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4} = 2$.

ⓑ arg(u) also draw Arg .



$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$\theta = \frac{\pi}{3}$.

$\arg = -\frac{\pi}{3}$.

ⓒ Polar form .

$1-\sqrt{3}i = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$.

ⓓ $(1-\sqrt{3}\,i)^3$

$(1-\sqrt{3}i)^3 = 2^{\overset{\swarrow}{3}}\left[\cos\left(-\frac{\pi}{3}\times \overset{\downarrow}{3}\right) + i\sin\left(-\frac{\pi}{3}\times \overset{\downarrow}{3}\right)\right]$

$= 8\left[\cos(-\pi) + i\sin(-\pi)\right]$.

$= 8\left[-1+0i\right] = -8$

ⓔ exponential form

$\rightarrow |z|e^{i(\arg)}$

$\Rightarrow 2e^{-i\frac{\pi}{3}}$.

ⓕ $|u|^4$

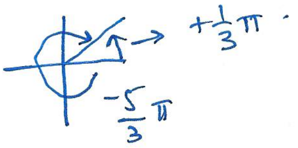
$= 2^4 = 2\times2\times2\times2 = 16$.

ⓖ $\arg(u^5)$

$= 5 \times -\frac{\pi}{3}$

$= -\frac{5}{3}\pi$

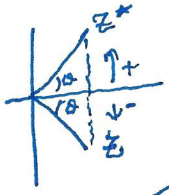
$=$ hence $\boxed{+\frac{1}{3}\pi}$.



magnitude of arg for conjugate is always same. sign is opposite.

As θ is reflected at x-axis vertically.

eg.



Note : If arg for z is $\frac{\pi}{6}$ then arg for z^6 is $\left(6 \times \frac{\pi}{3}\right) = 2\pi$ or "0"

VIMP → ⇒ angle from + x-axis

Q Find modulus & argument.

ⓐ $\frac{2}{z-1}$ where $z = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$

Sol

$$\frac{2}{\frac{1}{2}+\frac{1}{2}\sqrt{3}i - 1} = \frac{2}{\frac{1}{2}(\sqrt{3}i - 1)} = \frac{4}{\sqrt{3}i - 1} \rightarrow \text{Rationalise}$$

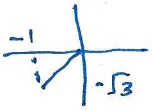
$$= \frac{4}{\sqrt{3}i - 1} \times \frac{-\sqrt{3}i - 1}{-\sqrt{3}i - 1}.$$

$$= \frac{-4\sqrt{3}i - 4}{-3i^2 - \sqrt{3}i + \sqrt{3}i + 1} = \frac{-4(\sqrt{3}i + 1)}{+3+1} = \frac{-4(\sqrt{3}i+1)}{+4}$$

= ~~2√3+2~~. $= -\sqrt{3}i - 1$

modulus ⇒ $\sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2.$

$\theta \Rightarrow \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$



arg ⇒ $= -\left(\pi - \frac{\pi}{3}\right)$.

$= -\frac{2}{3}\pi.$

Remember

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta.$

$\tan(-\theta) = -\tan\theta$

ⓑ $\frac{1}{(wz)^*}$ where $|z| = 3$, $\arg(z) = \frac{\pi}{4}$ and $|w| = 2$, $\arg(w) = \frac{\pi}{6}$

Sol

$$= \left[(wz)^*\right]^{-1} = \left[6\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)\right]^{-1} = \frac{1}{6}\left[\cos\left(-\frac{5\pi}{12}\right) - i\sin\left(-\frac{5\pi}{12}\right)\right]$$

$$= \frac{1}{6}\left[\cos\left(\frac{5\pi}{12}\right) + \sin\left(\frac{5\pi}{12}\right)\right]$$

mod ⇒ $\frac{1}{6}$, arg ⇒ $\frac{5\pi}{12}$.

Loci :-

$$|Z - 2 - 3i| \leqslant 1$$

ⓐ sketch
ⓑ largest $|Z|$
ⓒ least $|Z|$
ⓓ largest $\arg(Z)$
ⓔ least $\arg(Z)$

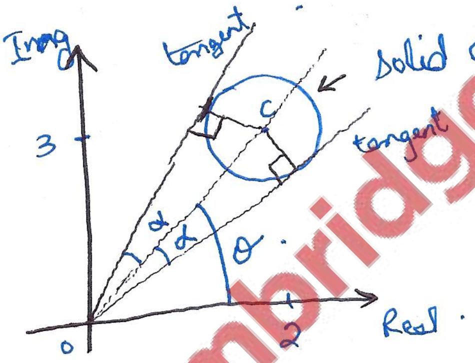
Note

↳ $|\underset{\substack{\downarrow\\ \text{every point}}}{Z} \overset{\text{must}}{-} \underbrace{(a+ib)}_{\text{centre}}| \leqslant \underset{\substack{\downarrow\\ \text{radius}}}{r}$

Sol

ⓐ $|Z - 2 - 3i| \leqslant 1$

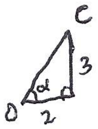
$|Z - (2+3i)| \leqslant 1$



solid circle cuz ⓢ → equal.

ⓑ $OC = \sqrt{3^2 + 2^2}$

$= \sqrt{13}$



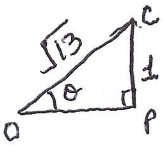
→ largest $|Z| =$ modulus + radius

$= \sqrt{13} + 1$.

ⓒ → least $|Z| =$ modulus − radius

$= \sqrt{13} - 1$.

ⓓ
ⓔ



$\theta = \sin^{-1}\left(\frac{1}{\sqrt{13}}\right) = 0.281$ rad.

$\alpha = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$ rad.

least arg $= \alpha - \theta = 0.983 - 0.281 = 0.702$ rad.

largest arg $= \alpha + \theta = 0.983 + 0.281 = 1.164$ rad.

Q

(i) On a sketch of an Argand diagram, show the locus representing complex numbers satisfying the equation

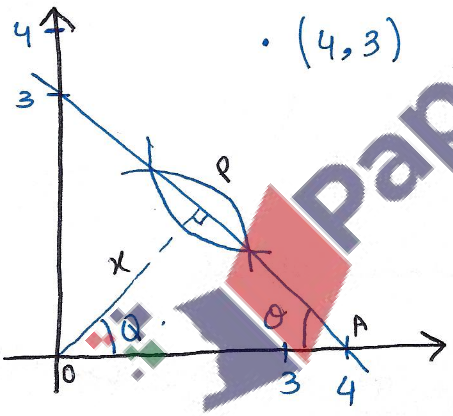
$$|z| = |z - 4 - 3i|$$

(ii) Find the complex number represented by the point on the locus where $|z|$ is least. Find the modulus and argument of this complex number, giving the argument correct to 2 decimal places.

Sol

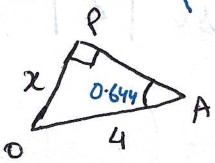
$$\left|z - \underbrace{(0+0i)}\right| = \left|z - \underbrace{(4+3i)}\right|$$

↳ points from where we have to draw an arc.



$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.644$ rad.

$Q = \pi - \left(\frac{\pi}{2} + 0.644\right) = 0.927$ rad.



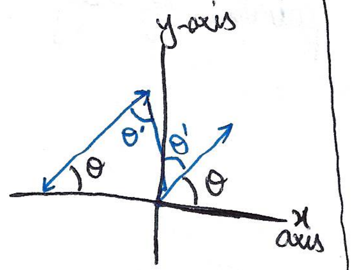
$\frac{x}{4} = \sin(0.644)$

least $|z| = x = 2.40$.

$\arg(\text{least } |z|) = 0.927$ rad.

hence

$$z = 2.4e^{0.927i}$$

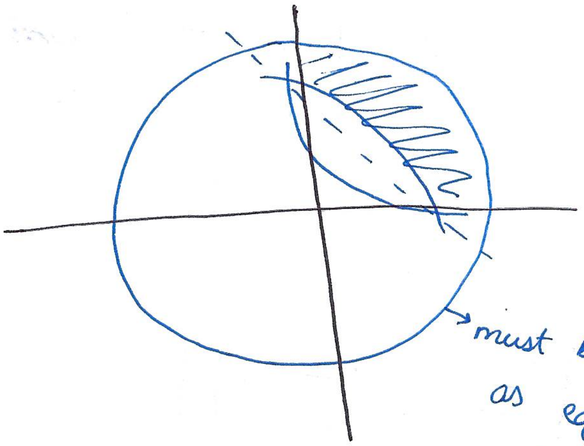


If θ' & θ' are equal (or) θ & θ are equal then segments are parallel.

Note

$$\underbrace{|Z-(0+2i)|}_{} < |Z-(-1-i)|$$

↳ must be inside shaded region

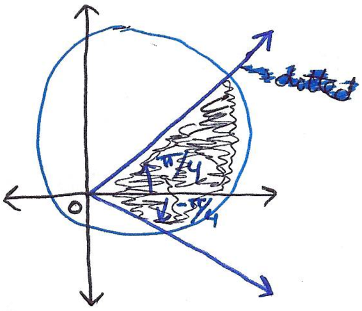


↳ must be (- - -) not solid as equal not present.

$|z-1-i| \leqslant 2$

$|z-(1+i)| \leqslant 2$

$-\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{4}$



Ray :

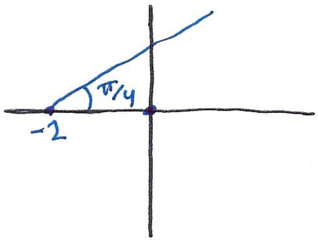
$\arg(Z+2) = \frac{\pi}{4}$

ⓐ sketch

ⓑ least $|Z|$ ← idea covered already

$\arg(Z-\underbrace{(-2+0i)}_{}) = \boxed{\frac{\pi}{4}}$ → angle with ⊕ x-axis.

→ starting point of ray.



Vectors :-

⊕ $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$ where O is origin

must

eg. $\overrightarrow{PK} = \overrightarrow{OK} - \overrightarrow{OP}$, $\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$

⊕ $\overrightarrow{OB} = 2i + j + k$ $\xrightarrow{\text{same}}$ $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ → x, → y, → z → axis

eg. $\overrightarrow{OA} = i + 2j + 3k \rightarrow \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Q: $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

ⓐ find $\overrightarrow{AB}$.

Sol

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

ⓑ unit vector of $\overrightarrow{AB}$.

⊕ unit vector $= \dfrac{\text{vector}}{\text{Magnitude}}$

eg. $\underbrace{\text{unit vector}}_{\hat{BC}}$ $\overrightarrow{BC} = \dfrac{\overrightarrow{BC}}{|\overrightarrow{BC}|}$

$$\hat{AB} = \frac{i - j + 2k}{\sqrt{(1)^2 + (-1)^2 + (2)^2}}$$

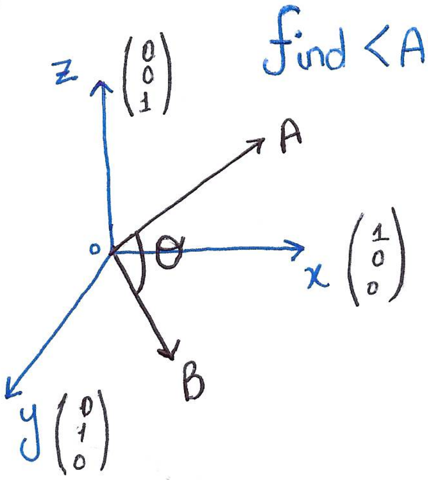
$$\hat{AB} = \boxed{\frac{i - j + 2k}{\sqrt{6}}}$$

(or)

$$= \boxed{\frac{1}{\sqrt{6}}i - \frac{1}{\sqrt{6}}j + \frac{2}{\sqrt{6}}k}$$

(or) $= \boxed{\hat{AB} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}$

Q



find <AOB

$\vec{OA} = \begin{pmatrix}2\\3\\1\end{pmatrix}$

$\vec{OB} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$

Diagram is not actually required.

⊕ Dot product

$\vec{OB} \cdot \vec{OA} = |\vec{OB}||\vec{OA}|\cos\theta$.

$\left(\begin{array}{ccc}3 & \rightarrow & 2\\ 1 & \cdot\rightarrow & 3\\ -1 & \rightarrow & 1\end{array}\right) = \sqrt{(3)^2+(1)^2+(-1)^2} \cdot \sqrt{(2)^2+(3)^2+(1)^2} \cdot \cos\theta.$

$6+3+(-1) = \sqrt{11} \cdot \sqrt{14} \cdot \cos\theta.$

$\frac{8}{\sqrt{11}\cdot\sqrt{14}} = \cos\theta.$

$\frac{8}{\sqrt{11\times14}} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{8}{\sqrt{154}}\right)$

$\theta = 49.9°.$

obtuse	$\theta > 90°$
acute	$\theta < 90°$
reflex	$180° < \theta < 360°$
right	$\theta = 90°$

$\vec{OA}\cdot\vec{OB} = 0$ then $\theta = 90°$ (Right)

$\vec{OA}\cdot\vec{OB} = +ve$ then $\theta < 90°$ (acute)

$\vec{OA}\cdot\vec{OB} = -ve$ then $\theta > 90°$ (obtuse)

Vector equation of line :-)

$\overrightarrow{OA} = \begin{pmatrix}1\\2\\2\end{pmatrix}$ —— $\overrightarrow{OB} = \begin{pmatrix}2\\3\\-4\end{pmatrix}$ — points on line AB BA -or

Sol :

starting point

$\gamma = \overrightarrow{OA} + \lambda \overrightarrow{AB}$ → direction of line

point on line

same

or

$\gamma = \overrightarrow{OB} + \lambda \overrightarrow{BA}$

same

$BA = \overrightarrow{OA} - \overrightarrow{OB}$

$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$

So Vector equation of line

$\gamma = \overrightarrow{OB} + \lambda \overrightarrow{BA}$

$\gamma = \begin{pmatrix}2\\3\\-4\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\6\end{pmatrix}$

Intersecting of a line

$r = \begin{pmatrix}1\\2\\0\end{pmatrix} + t\begin{pmatrix}-1\\5\\2\end{pmatrix}$ direction of line

point on line

$r = \begin{pmatrix}2\\-1\\5\end{pmatrix} + \mu\begin{pmatrix}3\\-2\\6\end{pmatrix}$

Sol Standard form.

$r = \begin{pmatrix}1-t\\2+5t\\2t\end{pmatrix}$, $r = \begin{pmatrix}2+3\mu\\-1-2\mu\\5+6\mu\end{pmatrix}$

If lines intersect :

$1 - t = 2 + 3\mu$ — ①

$2 + 5t = -1 - 2\mu$ — ②

$2t = 5 + 6\mu$ — ③

use any 2 equations & solve simultanously to get value of t & μ , ~~vero~~ verify the values of t & μ with 3rd equation (untouch) & if these values gives equal answers on both sides then lines interset .

$t + 3\mu = -1$ — ①

$5t + 2\mu = -3$ — ②

① × 5

$\cancel{5t} + 15\mu = -5$

$-\cancel{5t} \pm 2\mu = -3$

$13\mu = -2$

$\boxed{\mu = -\frac{2}{13}}$

put in ①

$t = -1 - 3\left(-\frac{2}{13}\right)$

$\boxed{t = \frac{-7}{13}}$

Check

put in ③

$2\left(\frac{-7}{13}\right) = 5 + 6\left(-\frac{2}{13}\right)$

$\frac{-14}{13} \neq \frac{53}{13}$ hence not intersect, lines skew

→ If Q asks for acute angle but you got 135° etc then acute angle is 180 − 135 = 45° ← Note

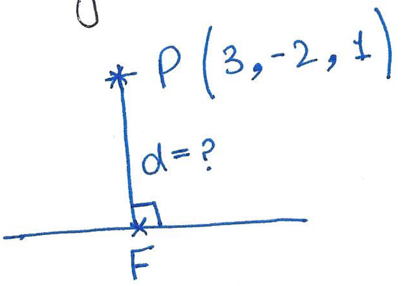
→ (length, distance) = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

→ Parallel lines have same "direction of line".

Q

The point "P" has a position vector $3i-2j+k$. The line l has equation $r=(4i+2j+5k)+\mu(i+2j+3k)$.

(i) Find the length of the perpendicular from "P" to "l", giving your answer correct to 3 significant figures.



Sol

$r = \begin{pmatrix}4\\2\\5\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}$ → direction of line.

Standard form of line:

$r = \overrightarrow{OF} = \begin{pmatrix}4+\mu\\2+2\mu\\5+3\mu\end{pmatrix}$

$\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP}$

$= \begin{pmatrix}4-\mu\\2+2\mu\\5+3\mu\end{pmatrix} - \begin{pmatrix}3\\-2\\1\end{pmatrix}$

$\overrightarrow{PF} = \begin{pmatrix}1-\mu\\4+2\mu\\4+3\mu\end{pmatrix}$

As $\overrightarrow{PF}$ $\boxed{\perp}$ → perpendicular to line.

$\overrightarrow{PF} \cdot b = 0$

$\begin{pmatrix}1-\mu\\4+2\mu\\4+3\mu\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 0$

$1-\mu+8+4\mu+12+9\mu=0$

$14\mu=-21$

$\mu = -\frac{3}{2}$

$\overrightarrow{PF} = \begin{pmatrix}-\frac{1}{2}\\1\\-\frac{1}{2}\end{pmatrix}$

$|\overrightarrow{PF}| = \sqrt{(-\frac{1}{2})^2+(1)^2+(-\frac{1}{2})^2}$

$d = \sqrt{\frac{1}{4}+1+\frac{1}{4}}$

$d = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$

$\frac{\sqrt{c}}{c} = \frac{1}{\sqrt{c}}$

$\frac{c}{\sqrt{c}} = \sqrt{c}$

Derivative of $\tan^{-1}(x)$:-

* $y = \tan^{-1}(ax)$

$$\frac{dy}{dx} = \frac{1}{1+(ax)^2} \times \frac{d}{dx}(ax)$$

Exp $y = \tan^{-1}(2x^2)$

$$\frac{dy}{dx} = \frac{4x}{1+(2x^2)^2} = \boxed{\frac{4x}{1+4x^4}}$$

Questions

ⓐ $y = \tan^{-1}(4x^2+2x+1)$

ⓑ $y = 4\tan^{-1}(\sqrt{2x})$

ⓒ $y = \tan^{-1}(1-2x)$

ⓓ $y = e^x\tan^{-1}(x)$

ⓔ $y = x^2.\tan^{-1}(x^2)$

Answers

ⓐ $y = \tan^{-1}(4x^2+2x+1)$

$$\frac{dy}{dx} = \frac{8x+2}{1+(4x^2+2x+1)^2}$$

ⓑ $y = \tan^{-1}(\sqrt{2x})$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(2x)^{-\frac{1}{2}} \times 2}{1+(\sqrt{2x})^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{2x}}}{1+2x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}(1+2x)}$$

ⓒ $y = \tan^{-1}(1-2x)$

$$\frac{dy}{dx} = \frac{-2}{1+(1-2x)^2}$$

ⓓ $y = e^x\tan^{-1}(x)$

$$\frac{dy}{dx} = e^x\left[\frac{1}{1+x^2}\right] + \tan^{-1}(x)e^x$$

$$= e^x\left[\frac{1}{1+x^2} + \tan^{-1}(x)\right]$$

V.IMP ☺

ⓔ $y = x^2\tan^{-1}(x^2)$

$$\frac{dy}{dx} = x^2\left[\frac{2x}{1+(x^2)^2}\right] + \tan^{-1}(x^2)2$$

$$= 2x\left[\frac{x^2}{1+x^4} + \tan^{-1}(x^2)\right]$$

Integration of $\frac{1}{a^2+x^2}$:-

* $$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\times\tan^{-1}\left(\frac{x}{a}\right)$$

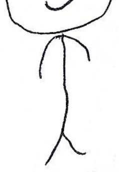
If $a=1$.

$$= \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x).$$

ⓐ $$\int \frac{1}{x^2+16}\,dx$$

$$= \int \frac{1}{x^2+4^2}\,dx$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$$



ⓑ $$\int \frac{1}{x^2+3}\,dx = \int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

Exp :-

① $\int \frac{1}{49+x^2}\,dx$

$= \int \frac{1}{7^2+x^2}\,dx$

$= \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)$

② $\int \frac{1}{1+49x^2}\,dx$

$= \int \frac{1}{1+(7x)^2}\,dx$

$= \frac{1}{7}\tan^{-1}(7x)$

③ $\int \frac{1}{1+5x^2}\,dx$

$= \int \frac{1}{1+(\sqrt{5})^2x^2}\,dx$

$= \int \frac{1}{1+(x\sqrt{5})^2}\,dx = \frac{\tan^{-1}(x\sqrt{5})}{\sqrt{5}}$

④ $\int \frac{1}{1+kx^2}\,dx$

$= \int \frac{1}{1+(x\sqrt{k})^2}\,dx$

$= \frac{\tan^{-1}(x\sqrt{k})}{\sqrt{k}}$

Solve

ⓐ $\int \frac{5}{25+x^2}\,dx.$ ⓑ $\int \frac{7}{x^2+81}\,dx$ ⓒ $\int \frac{1}{1+9x^2}\,dx.$

ⓓ $\int \frac{16}{6x^2+1}\,dx$ ⓔ $\int \frac{12}{9+16x^2}\,dx.$

Solution

ⓑ $7\int \frac{1}{x^2+9^2}\,dx.$

$= 7\times\frac{1}{9}\times\tan^{-1}\left(\frac{x}{9}\right)+c$

ⓓ $\int \frac{16}{6x^2+1}\,dx.$

$= 16\int \frac{1}{(\sqrt{6}x)^2+1}\,dx$

$= \frac{16\times\tan^{-1}(x\sqrt{6})}{\sqrt{6}}+c$

ⓐ $5\int \frac{1}{5^2+x^2}\,dx$

$= 5\times\frac{1}{5}\times\tan^{-1}\left(\frac{x}{5}\right)$

$= \tan^{-1}\left(\frac{x}{5}\right)+c$

ⓒ $\int \frac{1}{1+9x^2}\,dx$

$= \int \frac{1}{1+(3x)^2}\,dx.$

$= \frac{\tan^{-1}(3x)}{3}+c$

ⓔ $\int \frac{12}{9+16x^2}\,dx.$

$= 12\int \frac{1}{3^2+(4x)^2}\,dx$

$= 12\times\frac{1}{3}\tan^{-1}\left(\frac{4}{3}x\right)$

$= 4\tan^{-1}\left(\frac{4x}{3}\right)+c.$

* $\int \frac{6}{4+25x^2}\,dx.$

$= 6\int \frac{1}{2^2+(5x)^2}\,dx$

$= \frac{6}{2\times5}\tan^{-1}\left(\frac{5x}{2}\right)$

$= \frac{3}{5}\tan^{-1}\left(\frac{5x}{2}\right)+c.$

* $\int \frac{5}{16+7x^2}\,dx$

$= 5\int \frac{1}{4^2+(x\sqrt{7})^2}\,dx$

$= \frac{5}{4\cdot\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{7}}{4}x\right)+c$

* $\int \frac{3}{3+7x^2}\,dx$

$= 3\int \frac{1}{(\sqrt{3})^2+(x\sqrt{7})^2}\,dx.$

$= 3\times\frac{1}{\sqrt{3}}\times\frac{\tan^{-1}\left(\frac{x\sqrt{7}}{\sqrt{3}}\right)}{\sqrt{7}}$

$\Leftarrow = \frac{3}{\sqrt{3}\cdot\sqrt{7}}\times\tan^{-1}\left(\sqrt{\frac{7}{3}}x\right)$

$\frac{3}{\sqrt{21}}\times\tan^{-1}\left(\sqrt{\frac{7}{3}}x\right)$

$$* \int_0^5 \frac{20}{25+x^2}\,dx$$

$$= 20\int_0^5 \frac{1}{5^2+x^2}\,dx$$

$$= \left[20\times\frac{1}{5}\times\tan^{-1}\left(\frac{x}{5}\right)\right]_0^5$$

$$= 4\left[\tan^{-1}\left(\frac{5}{5}\right)-\tan^{-1}(0)\right]$$

$$= 4\left[\frac{\pi}{4}-0\right] = \pi .$$

$$* \int_{4\sqrt{3}}^{\frac{\sqrt{3}}{4}} \frac{8}{1+16x^2}\,dx$$

$$= 8\int \frac{1}{1+(4x)^2}\,dx$$

$$= \left[\frac{8\times\frac{1}{1}\tan^{-1}\left(\frac{4x}{1}\right)}{4}\right]_{4\sqrt{3}}^{\frac{\sqrt{3}}{4}}$$

$$= 2\left[\tan^{-1}(\sqrt{3})-\tan^{-1}(16\sqrt{3})\right]$$

= (-) 0.975 units.

Vectors :- Some missing important points :-

Notation :

- Vectors written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are said to be in column vector form.
- Vectors written as $xi + yj + zk$ are said to be in unit vector form.

Types of vectors :

- A displacement vector indicates a movement from one point to another.
- A position vector indicates a movement from the origin to a point.
- If A and B are points with position vectors "a" and "b" then $\overrightarrow{AB} = b - a$

☺ Straight lines :

- The vector equation of a straight line passing through a point with a position vector a and direction vector b is $r = a + tb$, where t is a scalar.
- Two lines are ~~intersect~~ parallel if they have the same direction but no point in common.
- Two lines intersect if they have one point in common.
- Two lines are coincident if they have the same direction and an infinite number of points in common.
- In three dimensions, skew lines that are not parallel but do not intersect.
- The angle between two straight lines is defined as the angle between their direction vectors.

Distance from a point to a line :

- To find the shortest distance from a point to a line, first find the coordinates of the foot of the perpendicular from the point to the line.

Arithmetic operations on $z_1 = a+bi$ and $z_2 = c+di$

- Addition : $z_1 + z_2 = (a+c) + (b+d)i$
- Subtraction : $z_1 - z_2 = (a-c) + (b-d)i$
- Multiplication : $z_1 z_2 = (ac-bd) + (ad+bc)i$
- Division : $\frac{z_1}{z_2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

$\boxed{i^2 = -1}$

$\boxed{x+iy} \rightarrow$ x & y are real values.

- Modulus : $|z| = \sqrt{x^2+y^2}$.
- Argument : Found using a diagram with $\tan\theta = \frac{y}{x}$.

$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$.

Polar forms

- Modulus - argument form : $r(\cos\theta + i\sin\theta)$.
- Exponential form : $re^{i\theta}$.

Differentiation All formulas

① $y = x^n$

$\frac{dy}{dx} = nx^{n-1}$

② $y = (2x^2+5)^{10}$

$\frac{dy}{dx} = 10(2x^2+5)^9 \times 4x$

$= 40x(2x^2+5)^9$.

③ $y = e^{5x}$ exponential

$\frac{dy}{dx} = e^{5x} \times \frac{d}{dx}(5x)$

$= e^{5x} \times 5$

④ $y = \sin(10x)$

$\frac{dy}{dx} = \cos(10x) \times \frac{d}{dx}(10x)$

$= 10\cos(10x)$

⑤ $y = \cos(20x^2)$

$\frac{dy}{dx} = -\sin(20x^2) \times \frac{d}{dx}(20x^2)$

$= -\sin(20x^2) \times 40x$

$= -40x\sin(20x^2)$

⑥ $y = \tan(3x)$

$\frac{dy}{dx} = \sec^2(3x) \times \frac{d}{dx}(3x)$

$= 3\sec^2(3x)$.

$6 = \ln e^6$

$k = \ln e^k$

$2 = \ln e^2$

$e^{\ln k} = k$

$e^{\ln 3} = 3$.

$a\ln b = \ln b^a$

⑦ $y = \ln(x^2+3)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2+3)}{x^2+3}$$

$$\frac{dy}{dx} = \frac{2x}{x^2+3}$$

⑧ $y = \sin^3(4x^2+1)$

$$\frac{dy}{dx} = 3\sin^2(4x^2+1) \times \cos(4x^2+1) \times 8x$$

$$= 24x\sin^2(4x^2+1) \times \cos(4x^2+1)$$

Power → base → Angle.

⑨ laws of log:

$\ln(x \cdot y) = \ln x + \ln y$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln x^y = y \cdot \ln x$

⑩ Product Rule :

$$\frac{d}{dx}(u.v) = \left[u \times \frac{d}{dx}(v)\right] + \left[v \times \frac{d}{dx}(u)\right]$$

⑪ Quotient Rule :

Sequence matter

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v \times \frac{d}{dx}(u)\right) - \left(u \times \frac{d}{dx}(v)\right)}{v^2}$$

- Note - gradient is zero when line is parallel to x-axis (or)

$\frac{dy}{dx}\ \left(\frac{1}{0}\right) \leftarrow$ (y-axis)

- $\ln(e^x) = x$.
- $e^{a(\ln x)} = x^a$.

⑫ Implicit function :

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(y^2) = 2y\left(\frac{dy}{dx}\right)$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(y^3) = 3y^2 \times \frac{dy}{dx}$$

$$\frac{d}{dx}(e^{3y}) = e^{3y} \times 3\frac{dy}{dx}.$$

find $\frac{dy}{dx}$ in terms of x & y

$$x^3 + xy^2 = 5x$$

differenciation :

$$3x^2 + x\left[2y' \frac{dy}{dx}\right] + y^2 = 5$$

$$3x^2 + 2xy\frac{dy}{dx} + y^2 = 5$$

$$\frac{dy}{dx} = \frac{5 - 3x^2 - y^2}{2xy}$$

Concept :- $0 \leq \theta \leq 360$

$\sin 2\theta = -0.3$

2 sinxcosx is wrong make sure you keep one in final equation.

sin cos tan

$2\theta = 180 + 17.45,$
$360 - 17.45,$
$540° - 17.45,$
$720 - 17.45,$

$\theta = 98.8, 171.3, 278.8,$
$351.3°.$

→ If $y = \tan^{-1}(x)$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

→ If $y = \tan^{-1}(ax)$, then, using the chain rule with $u = ax$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+(ax)^2} \times a$$

⑬ Parametric Equation :

target $\frac{dy}{dx}$.

$$\frac{d}{d\theta} = 1$$

$$\sin 2\theta = 2\sin\theta \cos\theta$$

$$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\cot\theta = \frac{1}{\tan}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$x = t^2 + 1$$

$$y = 2t + 3$$

$$\frac{dx}{dt} = 2t^1 + 0$$

$$\frac{dy}{dt} = 2 + 0$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2 \times \frac{1}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$\sin x \cos x = \boxed{\frac{1}{4}}$ X

↙ ↘

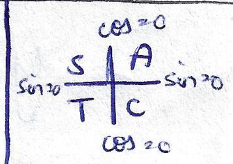
$\sin x = \frac{1}{4}$ $\cos x = \frac{1}{4}$

It should be zero to do so.

Chain Rule

x, y, ⓣ if θ then θ, and so on.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$



$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

P1 //

Chapter : 9 : Integration

"bad boy"

↳ Integration as the reverse of differentiation :

- If $\frac{d}{dx}\left[F(x)\right] = f(x)$, then $\int f(x)\,dx = F(x) + c$.

↳ Integration formulae :

- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ (where c is a constant and $n \neq -1$).
- $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$ ($n \neq -1$ & $a \neq 0$)

↳ Rules for indefinite integration :

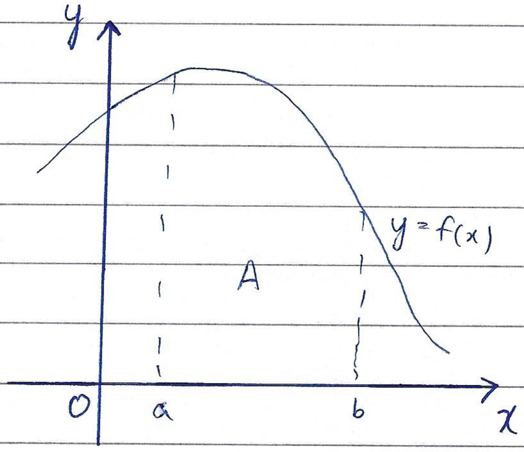
- $\int k f(x)\,dx = k\int f(x)\,dx$, where k is a constant.
- $\int \left[f(x) \pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx$

↳ Rules for definite integration :

- If $\int f(x)\,dx = F(x) + c$, then $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$.
- $\int_a^b k\,f(x)\,dx = k\int f(x)\,dx$, where k is a constant.
- $\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

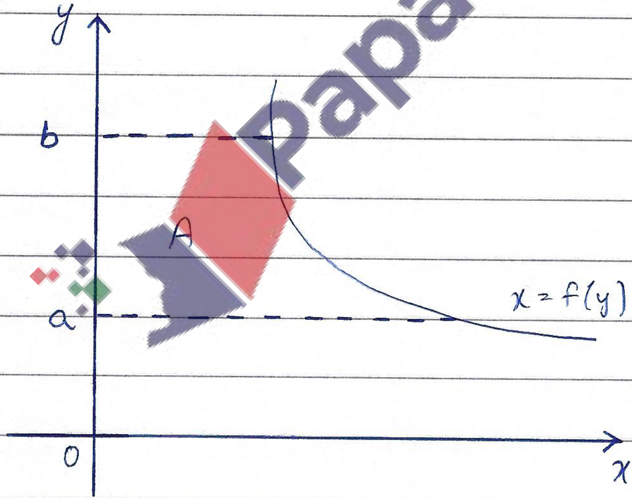
P2//

↳ Area under a curve :



- The area, A, bounded by the curve $y=f(x)$, the x-axis and the lines $x=a$ and $x=b$ is given by the formula:

$A = \int_a^b y\,dx$ when $\boxed{y \geq 0}$ (or $A = \int_a^b f(x)\,dx$) $\boxed{f(x) \geq 0}$

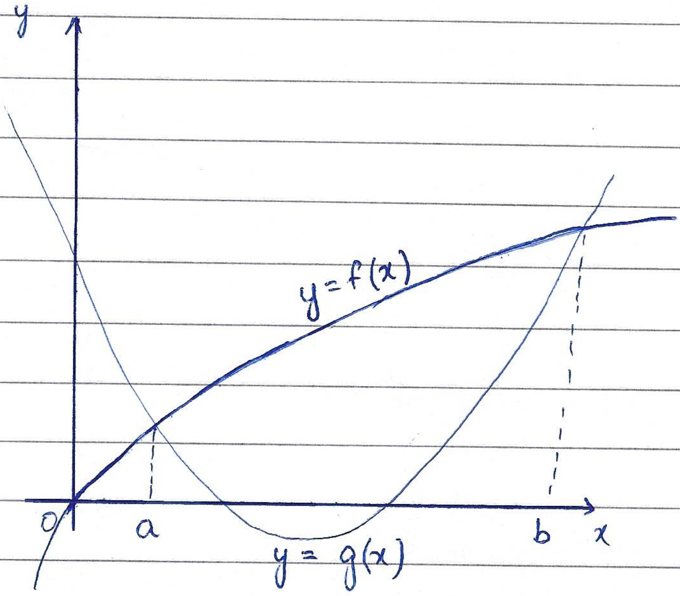


- The area, A, bounded by the curve $x=f(y)$, the y-axis and the lines $y=a$ & $y=b$ is given by the formula:

~~A~~ =

$A = \int_a^b x\,dy$ when $x \geq 0$ (or $A = \int_a^b f(y)\,dy$ when $f(y) \geq 0$)

P3.



- The area, A, enclosed between $y = f(x)$ and $y = g(x)$ is given by the formula:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

where a and b are the x-coordinates of the points of intersection of the functions f and g.

↳ Improper integrals:

- Integrals of the form $\int_a^\infty f(x)\,dx$ can be evaluated by replacing the infinite limit with a finite value, X, and then taking the limit as $X \to \infty$, provided the limit exists.

Py

- Integrals of the form $\int_{-\infty}^{b} f(x)\,dx$ can be evaluated by replacing the infinite limit with a finite value, X, and then taking the limit as X → −∞, provided the limit exists.

- Integrals of the form $\int_{a}^{b} f(x)\,dx$ where $f(x)$ is not defined when $x=a$ can be evaluated by replacing the limit a with an X and then taking the limit as X → a, provided the limit exists.

- Integrals of the form $\int_{a}^{b} f(x)$ is not defined when $x=b$ can be evaluated by replacing the limit b with an X and then taking the limit as X → b, provided the limit exists.

⟷

↳ Volume of Revolution:

- The volume, V, obtained when the function $y=f(x)$ is rotated through 360° about the x-axis b/w the boundary values $x=a$ and $x=b$ is given by the formula $V=\int_{a}^{b} \pi y^2\,dx$.

- The volume, V, obtained when the function $x=f(y)$ is rotated through 360° about the y-axis b/w the boundary values $y=a$ and $y=b$ is given by the formula $V=\int_{a}^{b} \pi x^2\,dy$.

⟷

Solve Past papers.

Chapter : 1

Representation of Data

Raw data: 2, 7, 6, 1, 3

Un-grouped :

Score	2	3	5
frequency	5	1	1

Group :

marks	0-10	11-20	21-50
frequency	5	3	2

Examples : ① example :-

3, 4, 4, 8, 10

→ ⓐ mean $\bar{x} = \frac{\sum x}{n} = \frac{3+4+4+8+10}{5} = 5.8$

→ ⓑ mode most repeating number - ④

→ ⓒ median data must be ordered

$\text{median} = \frac{n+1}{2} = \frac{5+1}{2} = 3^{rd} \rightarrow (4)$ either side

→ ⓓ standard deviation (σ) S.D

$$\sigma = S.D = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}$$

$$= \sqrt{\frac{3^2+4^2+4^2+8^2+10^2}{5} - (5.8)^2}$$

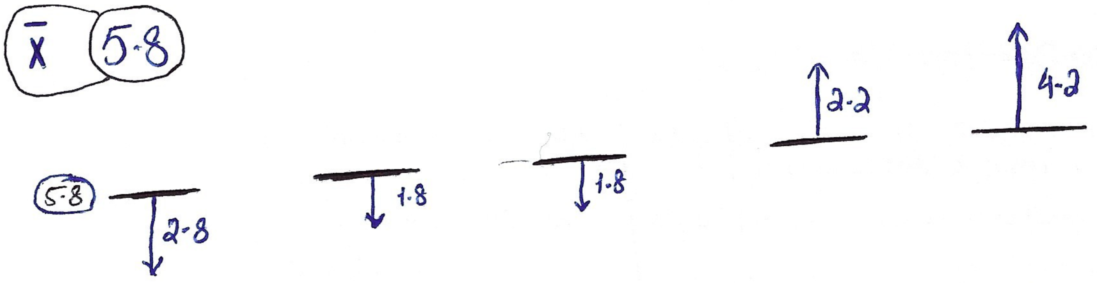
$$= \sqrt{7.36} = 2.71$$

→ ⓔ Variance (σ^2)

$$\sigma^2 = \frac{\sum x^2}{n} - (\bar{x})^2$$

$$= (\sqrt{7.36})^2 = 7.36.$$

3 4 4 8 10



deviation will be close to "0" if each value of x is close to mean $\bar{x}$.

less deviation (S.D) , more consistant

② example :-

Marks	frequency	mid-class (x)	fx	fx²
0-10	5	$\frac{0+10}{2}$ 5	5x5 25	25x5 125
10-20	6	$\frac{10+20}{2}$ 15	6x15 90	90x15 1350
20-30	7	$\frac{20+30}{2}$ 25	25x7 175	175x25 4375

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{25+90+175}{5+6+7} = 16.1$$

$$(\sigma) = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2}$$

$$= \sqrt{\frac{125+1350+4375}{5+6+7} - (16.1)^2}$$

$$= 8.1 .$$

Find mean & S.D by coding method :

13, 14, 15

Solution :

a = assumed mean

let $a = 14$ → same answer if you take 13, 15.

x	$x-a$	$(x-a)^2$
13	-1	1
14	0	0
15	1	1
	$\Sigma(x-a) = 0$	$\Sigma(x-a)^2 = 2$

$$\Rightarrow \boxed{\bar{x} = \frac{\Sigma(x-a)}{n} + a}$$

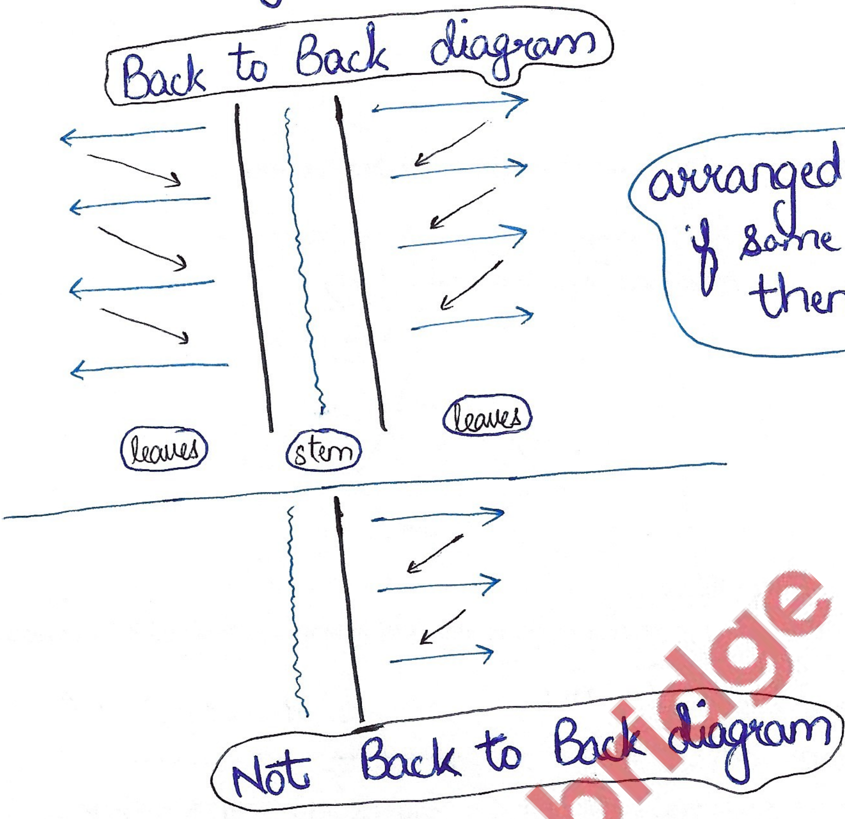
$$\bar{x} = \frac{0}{3} + 14 = 14.$$

$$\Rightarrow \boxed{\sigma = \sqrt{\frac{\Sigma(x-a)^2}{n} - \left[\frac{\Sigma(x-a)}{n}\right]^2}}$$

$$= \sqrt{\frac{2}{3} - \left(\frac{0}{3}\right)^2}$$

$$= 0.816$$

Stem and leaves diagram :



draw with pencil first

arranged in order if same value, show them as well

P6 - Oct - 2008.

Example :-

115 120 158 132 125 104 142 160 145 104

162 117 109 124 134

Draw stem & leaf diagram : **Step 1 :** ① smallest value : 104

② highest value : 162 → skip

10 → 16 .

Step 2 :

10	4	4	9
11	5	7	
12	0	4	5
13	2	4	
14	2	5	
15	8		
16	0	2	

Step 3 : 15 | 8 means 158 etc (Beats/min)

↑ any set from diagram .

③ **Median** = $\frac{n+1}{2} = \frac{15+1}{2} = 8^{th} = 125$

Quantiles means upper & lower quantiles

④ **U.Q : Upper Quantiles** = $\frac{3}{4}(n+1)$

$= \frac{3}{4}(15+1) = 12^{th}$

$= 145$

interquantile range IQR = U.Q - L.Q

⑤ **L.Q : Lower Quantiles** = $\frac{1}{4}(n+1)$

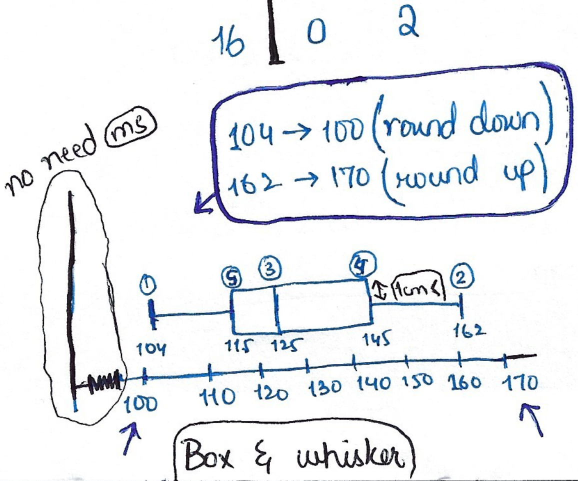
$= \frac{1}{4}(15+1) = 4^{th}$

$= 115 .$

125 - 115 = 10 = IQR

104 → 100 (round down)

162 → 170 (round up)



Box & whisker

Draw histogram :

data must be continuous data.

if not continous, discontinuous, convert them into continous.

1-10 → same
10-20 → same means continuous
20-30

Marks	f	class Boundries	frequency density
1-10	20	0.5 —— 10.5	$20 \div (10.5-0.5) = 2$
11-30	20	10.5 —— 30.5	1
31-40	20	30.5 —— 40.5	2
41-50	30	40.5 —— 50.5	3.

→ not same so convert it into continuous → "class boundries" → additional step.
(discontinuous)

class boundries :

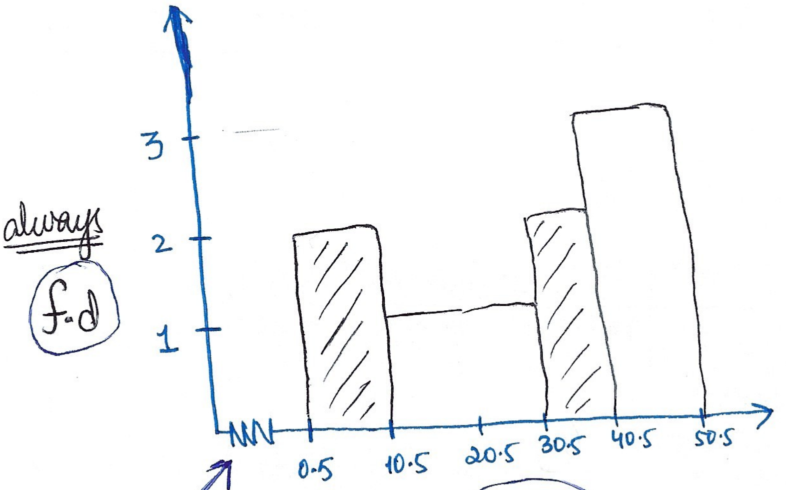
1 —— 10
11 —— 30
↑ lower limit ↑ upper limit

Take difference and divide by 2.

$\frac{(11-10)}{2} = 0.5$

- * Add 0.5 to every upper limit and subtract 0.5 to every lower limit
- * We make class boundries to convert discontinuous data to continous data.
- * Frequency density (f.d) = $\frac{\text{frequency (f)}}{\text{class width (c.w)}}$

Class width = (upper limit − lower limit) of class boundries.



area of column represents frequency.

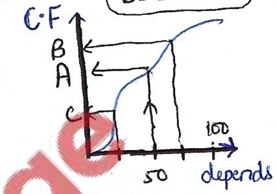
→ values must match class boundaries

allowed on x & y-axis

Marks

continuous data always.

IGCSE Maths



B-C = IQR

B → UQ
C → LQ

For median, go up from 50 "A" is the answer etc

⇒ Box & whisker :-

Advantages :
- Handles large data easily
- A clear summary
- Displays abnormal results clearly.

Disadvantages :
- exact values (not retained) (lost).

⇒ Stem & leaf :-

Advantages :
- We can find median Quartile very easily
- It can quickly organise data.

Disadvantages :
- It is not effective for large data.
- It is not giving summary.

⇒ Histogram :-

Advantages :
- It provides a way to display the frequency of occurrences of data along intervals.

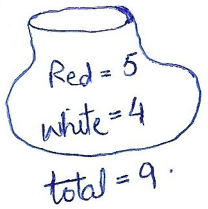
Disadvantages :
- Use of intervals the calculation of exact measure of central tendency

↳ mean, mode, median

Chapter : 2

Probability

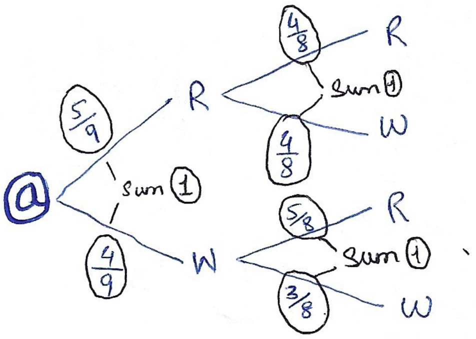
Question :-



withdraw 2 balls without replacement.

ⓐ Draw tree diagram

ⓑ Find probability



ⓘ 2nd ball Red

ⓘⓘ 1st ball white $\overset{\boxed{\text{AND}}}{\underline{\text{given that}}}$ 2nd Ball Red (conditional probability)

Note

* In branch or process always multiply
* More than one branches, add up.

AND ⇒ multiply

OR ⇒ add.

ⓑⓘ $P(\text{2nd ball Red}) = \left(\frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{4}{9} \times \frac{5}{8}\right) = \frac{5}{9}$.

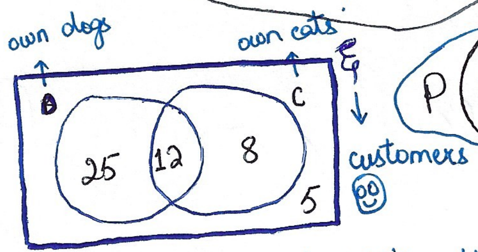
ⓘⓘ $P = \frac{\text{common probability b/w before \& after "given that"}}{\text{Probability after "given that"}}$.

$$= \frac{\frac{4}{9} \times \frac{5}{8}}{\frac{5}{9}} = \frac{1}{2}.$$

Concept :

conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{12/50}{37/50} = \frac{12}{37}$$ → answer

⇒ What is the propability that a customer, chosen at random, owns a cat, given that they also owns a dog.

Concept :

- Binomial probability distribution

$B(n, p)$

n → total objects

p → probability of favor

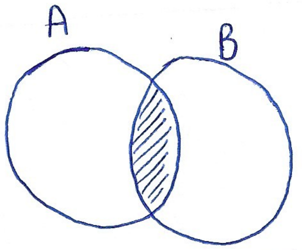
$\boxed{q = 1 - p}$

Normal distributions

$N(\mu, \sigma^2)$

μ → mean

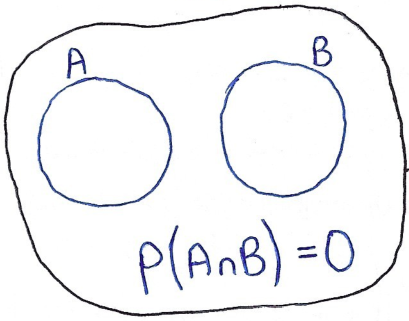
σ^2 → variance



$p(A \cap B)$

$p(A \text{ and } B)$

Same answer



* → A and B are mutually exclusive events/sets.

* $p(A) \times p(B) = p(A \cap B)$ ← not conditional probability

if true ↑ then A & B are independent.

Chapter : 3

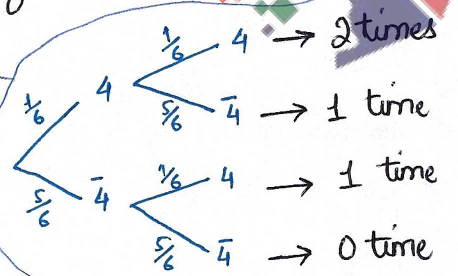
Discrete Random Variable

Question :

one fair dice is rolled twice. "X" represents that dice lands on "4".

ⓐ find propability distribution table ⓑ find $\bar{x}$ ⓒ find σ.

ⓐ



Propability distribution table

X	0	1	2
$p(X=x)$	$\frac{5}{6} \times \frac{5}{6}$	$2 \times \frac{1}{6} \times \frac{5}{6}$	$\frac{1}{6} \times \frac{1}{6}$
$p(X)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

same

Note $\frac{25}{36} + \frac{5}{18} + \frac{1}{36} = 1$ //

If true then table is correct.

ⓑ mean $= \bar{x} = E(x) = \mu =$ expected value (Expectation)

pastpaper might use these

$\frac{\Sigma f x}{\Sigma f}$ → alway ① → sum of $\boxed{P(x)}$

$$\bar{x} = \left(0 \times \frac{25}{36}\right) + \left(1 \times \frac{5}{18}\right) + \left(2 \times \frac{1}{36}\right) = \frac{1}{3} //$$

additional info

ⓒ $\boxed{\sigma^2 = \text{Variance}}$

$$\sigma = \sqrt{\left(0^2 \times \frac{25}{36}\right) + \left(1^2 \times \frac{10}{36}\right) + \left(2^2 \times \frac{1}{36}\right) - \left(\frac{1}{3}\right)^2} = 0.527 //$$

Binomial probability distribution :

$$\boxed{{}^nC_r\, q^{(n-r)}\, p^r}$$

p is propability of favor

q is propability of not favor.

n is total objects

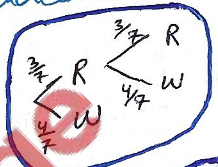
r is taken at a time object .

When n is very large

Conditions

all conditions must be present at the same time .

① – constant p (independent propability) → when replacement is there .



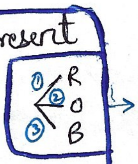
① ②

② – fixed numbers of trials (finite trial) .

X

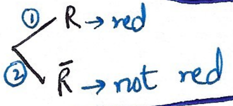
③ – only two outcomes

If R, B, O present then instead of



→ 3 outcomes X .

write



$\mu = n\,p$

μ → mean

n → total observation

p → propability of favor

$\sigma^2 = n\,p\,q$

σ² → variance

q → propability of not favor.

Question :

→ 3 dice roll

- 2 dice lands on 5

$n = 3$ → 3 dice

$r = 2$

$p = \frac{1}{6}$

$q = \frac{5}{6}$

$${}^3C_2\left(\frac{5}{6}\right)^{3-2}\left(\frac{1}{6}\right)^2 = \frac{5}{72}$$

- 3 dice lands on 5

$n = 3$

$r = 3$

$p = \frac{1}{6}$

$q = \frac{5}{6}$

$${}^3C_3\left(\frac{5}{6}\right)^{3-3}\left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

- 1 dice lands on 5

$n = 3$

$r = 1$

$p = \frac{1}{6}$

$q = \frac{5}{6}$

$${}^3C_1\left(\frac{5}{6}\right)^{3-1}\left(\frac{1}{6}\right)^1 = \frac{25}{72}$$

Selection / chosen

means no replacement ,,

Question:

$B\left(6, \frac{1}{3}\right) \qquad p(x \leq 5)$

X

$(r=0)+(r=1)+(r=2)+$
$(r\neq3)+(r\neq4)+(r\neq5)$

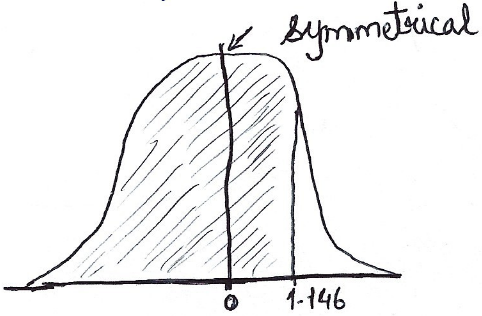
$[1-(r=6)]$ ✓

r = 6

$$1-\left[{}^{6}C_{6}\left(\frac{2}{3}\right)^{6-6}\left(\frac{1}{3}\right)^{6}\right] = 1-\left[\frac{1}{729}\right] = \frac{728}{729}$$

Chapter: 4

- ① Area = propability
- ② Table gives us values from left of bells shape
- ③ It will give value from left side only.



+

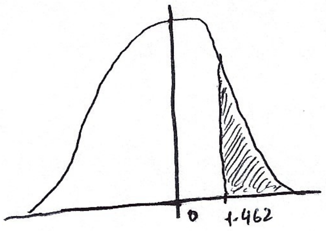
$P(Z < 1.146)$

$\phi(1.146)$

0.8741

$$\begin{array}{r} 0.8729 \\ +\ 0.0012 \\ \hline 0.8741 \end{array}$$

+

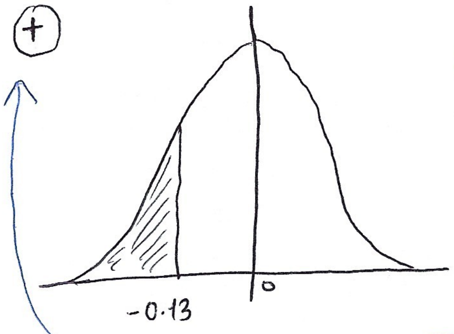


$P(Z > 1.462)$

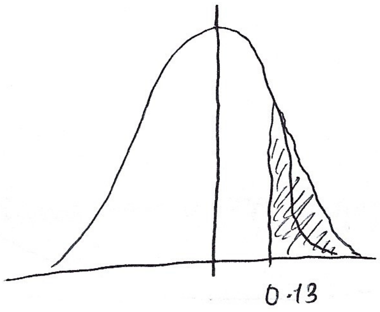
$1-\phi(1.462)$

$1-0.9282$

0.0718.



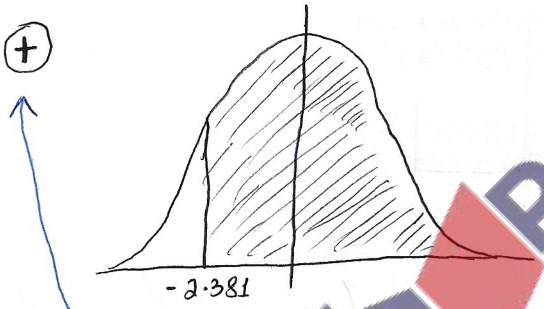
Same



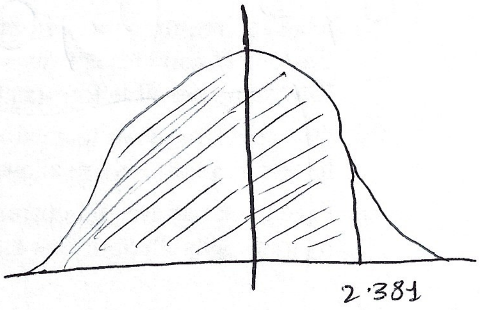
$$P(Z < -0.13)$$
$$= \phi(-0.13)$$
$$= 1 - \phi(+0.13)$$
$$= 1 - 0.5517$$
$$= 0.4483.$$

$$Z = \frac{X - \mu}{\sigma}$$

"Z"	"X"
"no need to convert"	"We have to convert".



Same



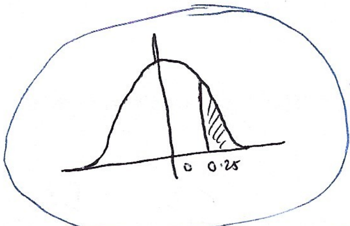
$$P(Z > -2.381)$$
$$= \phi(2.381)$$
$$= 0.9913$$

⊕ $N(2.5, 36)$

$P(~~Z~~\ X > 4)$

$= P\left(Z > \frac{4-2.5}{6}\right)$

$= P(Z > 0.25)$ →



not compulsory.

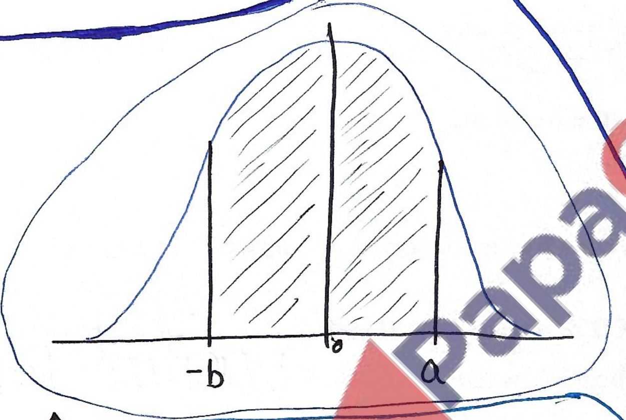
$= 1 - \Phi(0.25)$

$= 1 - 0.5987$

$= 0.4013.$

Rules

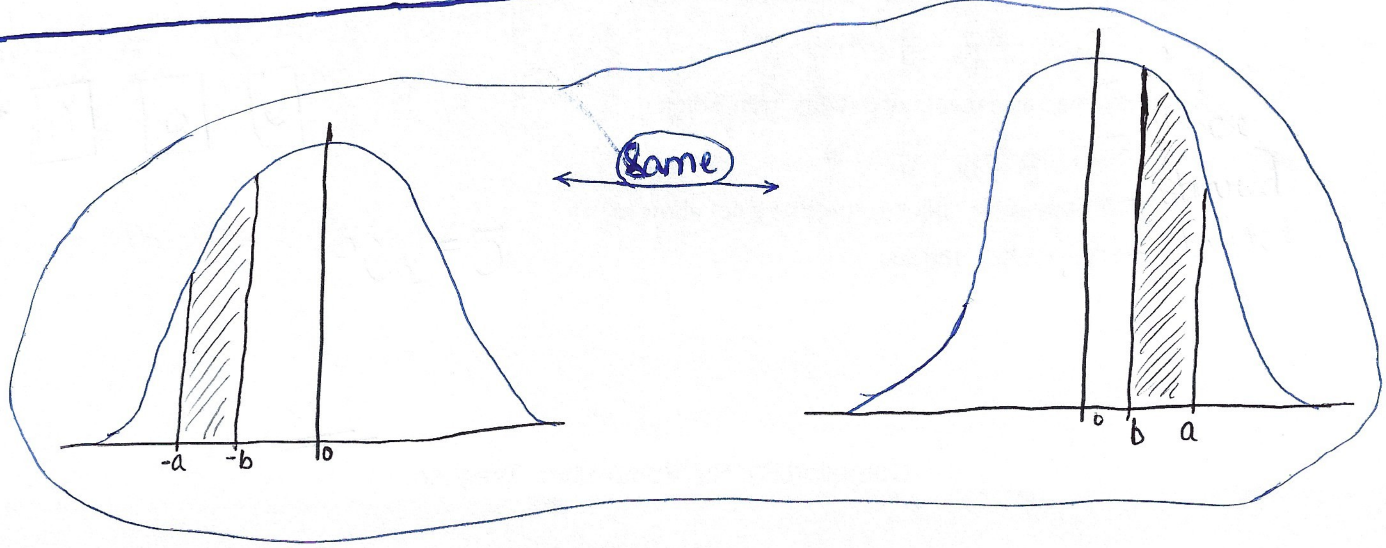
⊖



$\Phi(a) + \Phi(b) - 1$

$\Phi(a) - \Phi(b)$

⊖



Chapter : 5

Permutation, Combination

Permutation :

- * When order is important

eg $\boxed{32 \ \& \ 23}$

total objects

$^{n}P_{r}$

taken at a time

Example :

⇨ $\boxed{A}$ $\boxed{B}$ $\boxed{C}$ taken 2

AB
AC
BA
BC
CA
CB

$\boxed{^{3}P_{2} = 6}$

⇨ 1, 2, 3, 4

3 digit #

n = 4
r = 3

$\boxed{^{4}P_{3} = 24}$

Combination :

- * When order is not important

eg $\boxed{22 \ \& \ 22}$

$^{n}C_{r}$

- Select
- choices
- choose
- shared
- divided

* Key words which gives a hint.

Example :

⇨ $\boxed{A}$ $\boxed{B}$ $\boxed{C}$ taken 2

AB
AC
BC

$\boxed{^{3}C_{2} = 3}$

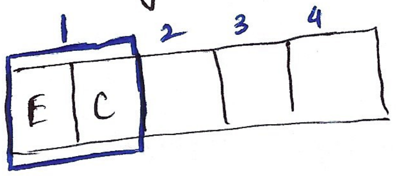
Note

AB & BA is considered same.
AC & CA " "
BC & CB " "

Example :

⇒ A, B, C, D, E

ⓐ E sitting next to C .



$^4P_4 \times {}^2P_2 = 48$.

ⓑ B refuses next to D .

when B & D sit together = 48 .

Total options = $^5P_5 = 120$

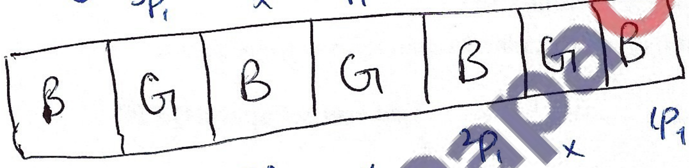
Don't sit together = 120-48 = 72.

⇒ 7 children throws a ball .

3G 4B .

ⓐ Boys & Girls alternatively

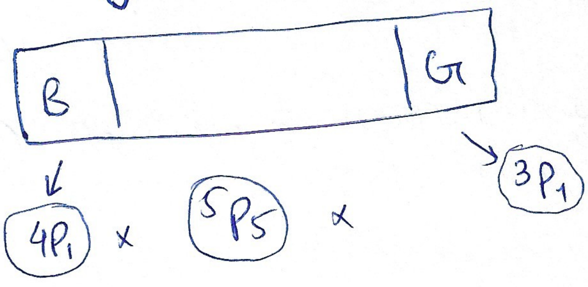
$^3P_1 \times {}^2P_1 \times {}^1P_1 = {}^3P_3$



$^4P_1 \times {}^3P_1 \times {}^2P_1 \times {}^1P_1 = {}^4P_4$

$^3P_3 \times {}^4P_4 = 144$.

ⓑ first by B & last by G .



$^4P_1 \times {}^5P_5 \times {}^3P_1 = 1440$.

Remember

Note

* Repeated letter
+
Combination
Special case .

May 2012 , V3

→ INCLUDE

Total = 7
Vowel = 3
Consonant = 4.

ⓐ all consonant together

C	C	C	C			
1				2	3	4

$4P_4 \times 4P_4 = 576$.

ⓑ ~~C & V are not next~~
arranged alternatively

	$3P_1$ ×		$2P_1$ ×		$1P_1$	
C	V	C	V	C	V	C
$4P_1$ ×		$3P_1$ ×		$2P_1$ ×		$1P_1$

$= 3P_3$ × $= 4P_4$

$3P_3 \times 4P_4 = 144$.

→ 1, 2, 2

$= \frac{3P_3}{2P_2} = 3$

$n P_n = n!$
↑ Same ↑ →↑
Concept

→ AAPPPLE.

$\frac{7P_7}{2P_2 \times 3P_3} = 420$.

→ 1, 2, 3, 4, 5, 6

ⓐ 3 digit #

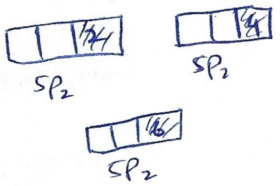
$6P_3 = 120$.

Ⓞ

$6P_1 \times 5P_1 \times 4P_1 = 120$.

$\boxed{x P_1 = x}$

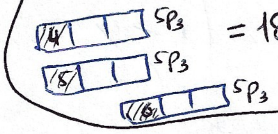
ⓑ 3 digit even #



$5P_2 + 5P_2 + 5P_2 = 60$

Ⓞ $5P_2 \times 3P_1 = 60$.

ⓒ 4 digit > 4000



$= 180$.

→ 5 Womens & 6 ~~Mens~~ Mens. 4 people chosen.

ⓐ No restriction $\boxed{11C_4}$

ⓑ atleast one women. $6C_3 \times 5C_1 + 6C_2 \times 5C_2 + 6C_1 \times 5C_3 + 6C_0 \times 5C_4 = 315$.

ⓒ One men ~~&~~ women are husband wife. find options where one must included but not both.

4 ← 5 womens
1 → husband wife.
1 ← 6 Mens
5 ← 6

when husband confirmed

$1C_1 \times 9C_3 = 9C_3$

when wife confired

$1C_1 \times 9C_3 = 9C_3$

$9C_3 + 9C_3 = 168$

New Chapter

⇒ Chapter 8 → S1 Cambridge book.

Must Solve ↗

A2 Maths

* Probability = [redacted] $\boxed{q^{(r-1)} \times p}$, where success is on r^{th} term.

* $Geo(0.2) \Rightarrow p = 0.2$, $q = 0.8$

ⓐ $P(X=1) = 0.2$ ⓑ $P(X=2) = (0.8)^{2-1} \times 0.2 = 0.16$.

ⓒ $P(X=3) = (0.8)^{3-1} \times 0.2 = 0.128$.

ⓓ $P(X \leq 3) = (X=1) + (X=2) + (X=3) = 0.2 + 0.16 + 0.128 = 0.488$.

* $P(X > r) = q^r$

ⓔ $P(X \leq 2) = (X=1) + (X=2) = 0.2 + 0.16 = 0.36$.

$\left[1 - P(X > 2)\right] = 1 - 0.8^2 = 0.36$.

$E(X) = \frac{1}{p}$.

* $X \sim Geo(0.8)$

$p = 0.8$, $E(X) = 1.2$

$P[X < E(X)] = P[X < 1.2]$

$= (X=1) = 0.8$

* $X \sim Geo(0.4) \rightarrow p = 0.4$, $q = 0.6$.

$$P(X=6 \mid X>4) = \frac{P(X=6 \text{ and } X>4)}{P(X>4)}$$

$$= \frac{P(X=6)}{P(X>4)} \leftarrow \text{common value.}$$

$$= \frac{0.6^5 \times 0.4}{0.6^4} = 0.6 \times 0.4 = 0.24.$$

Remember

Conditions for geometric distribution

ⓐ Constant probability

ⓑ finite trial

ⓒ Independent probability

Same for binomial

$\boxed{{}^nC_r \, q^{n-r} \, p^r}$

Topic: Probability

The **probability** of an event is a measure of the likelihood it will happen.

A probability of 0 indicates that the event is **impossible**.

A probability of 1 (or 100%) indicates that the event is **certain** to happen.

All other events have a probability between 0 and 1.

Equally likely outcomes

$$P(A) = \frac{n(A)}{n(S)}$$

← number of outcomes in event A

← total number of outcomes in the possibility space S

Complement

The complement of A is A' where A' is the event 'A does not occur'.

$P(A') = 1 - P(A)$

Combined events

For events A and B,

$$P(A \textbf{ or } B) = P(A) + P(B) - P(A \textbf{ and } B)$$

Remember that '**or**' means A or B or both.

In set notation $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Mutually exclusive events

For mutually exclusive events A and B

$$P(A \text{ and } B) = 0,$$

so $\quad P(A \text{ or } B) = P(A) + P(B)$ $\quad$ '**or**' rule for mutually exclusive events

In set notation $\quad P(A \cup B) = P(A) + P(B)$

Conditional probability

$$P(A, \text{given } B) = P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

so $\quad P(A \text{ and } B) = P(B) \times P(A \mid B) = P(A) \times P(B \mid A)$

Independent events

For independent events A and B

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

$P(A \text{ and } B) = P(A) \times P(B)$ $\quad$ '**and**' rule for independent events

In set notation $\quad P(A \cap B) = P(A) \times P(B)$

Topic: Discrete Random Variable

A list of all possible values of the discrete random variable X, together with their associated probabilities, is called a **probability distribution.**

The **sum** of the probabilities of all possible values of a discrete random variable X is 1.

i.e. $\sum P(X = x) = 1$ or $\sum p = 1$ 			p is shorthand for $P(X = x)$

E(X), the expectation (mean, expected value) of X

$$\mu = E(X) = \sum xp$$

Var(X), the variance of X

$$\sigma^2 = \text{Var}(X) = \sum (x - \mu)^2 p$$

Alternative version

$$\sigma^2 = \text{Var}(X) = \sum x^2 p - \{E(X)\}^2 \qquad \text{i.e. } \text{Var}(X) = \sum x^2 p - \mu^2$$

Topic: Binomial Distribution

If X is the number of successful outcomes in n independent trials and p is the probability of a successful outcome, then $X \sim B(n, p)$.

$$P(X = r) = \underset{\text{Number of ways to choose } r \text{ from } n}{\binom{n}{r}} \underset{\text{Probability of } r \text{ successes}}{p^r} \underset{\text{Probability of } (n - r) \text{ failures}}{q^{n-r}} \qquad \text{where } q = 1 - p, \qquad \text{for } r = 0, 1, 2, \ldots, n$$

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n - r)!}$$

Expectation $E(X) = \mu = np$

Variance $\text{Var}(X) = \sigma^2 = npq$

Standard deviation $\sigma = \sqrt{npq}$

The most likely number of successes (the mode) is the value of X with the highest probability.

Topic: Normal Distribution

Standard normal variable Z

$Z \sim N(0, 1)$ mean = 0, variance = 1, standard deviation = 1

Normal variable X

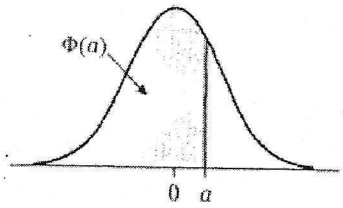
$X \sim N(\mu, \sigma^2)$ mean = μ, variance = σ^2, standard deviation = σ

To **standardise** X, use $Z = \dfrac{X - \mu}{\sigma}$

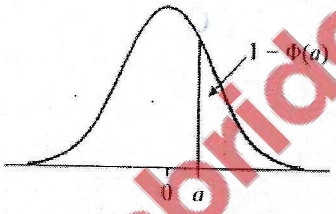
To find **probabilities** use the **normal distribution table**

In these illustrations $a > 0$, $b > 0$, $a < b$

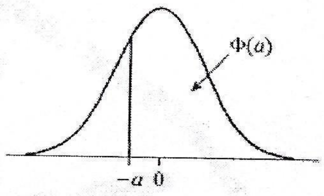
$P(Z < a) = \Phi(a)$



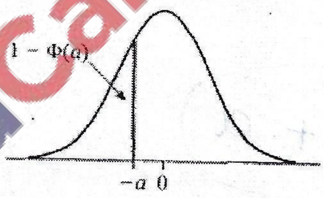
$P(Z > a) = 1 - \Phi(a)$



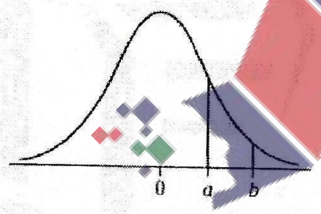
$P(Z > -a) = \Phi(a)$



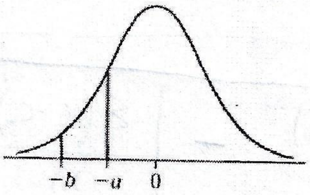
$P(Z < -a) = 1 - \Phi(a)$



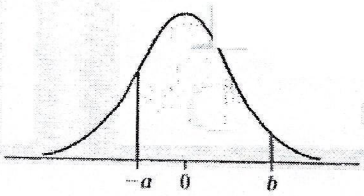
$P(a < Z < b) = \Phi(b) - \Phi(a)$



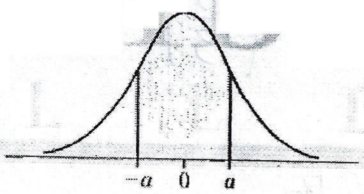
$P(-b < Z < -a) = \Phi(b) - \Phi(a)$



$P(-a < Z < b) = \Phi(b) - (1 - \Phi(a))$



$P(-a < Z < a) = 2\Phi(a) - 1$

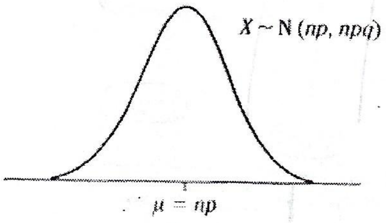


To find z values, read the normal distribution table in reverse, or use the table of **critical values** (page 216).

If $\Phi(z) = p$, i.e. $P(Z < z) = p$, then $z = \Phi^{-1}(p)$

The normal approximation to the binomial distribution

If $X \sim \mathrm{B}(n, p)$ and $np > 5$ and $nq > 5$, where $q = 1 - p$
then $X \sim \mathrm{N}(np, npq)$ with $\mu = np$ and $\sigma = \sqrt{npq}$



Continuity corrections

Continuity corrections must be used when calculating binomial probabilities using a normal approximation, for example

$P(X < 32) \rightarrow P(X < 31.5)$ $P(X \leq 32) \rightarrow P(X < 32.5)$
$P(X > 32) \rightarrow P(X > 32.5)$ $P(X \geq 32) \rightarrow P(X > 31.5)$
$P(3 < X < 10) \rightarrow P(3.5 < X < 9.5)$ $P(3 \leq X \leq 10) \rightarrow P(2.5 < X < 10.5)$

Topic: Representation of Data

Measures of central tendency (averages)

	Advantages	Disadvantages
Mode	It is useful when the most popular category is needed, for example clothes or shoe sizes.	It is not useful in very small data sets or when there are more than two modes. There may not be a mode. It may not be representative, for example it could be the lowest value. The modal class depends on the grouping of the data. It is not useful for further analysis.
Median	It is not affected by extreme values. It can be found as soon as a middle value is known, such as the distribution of times of runners in a race.	It does not use the whole data set. It is not useful for further analysis.
Mean	It is calculated using all the data and so represents every item. It is calculated using a mathematical formula, so calculators can be programmed to find it. It is extremely useful for further analysis.	It can be unduly affected by one or two extreme values.

Measures of variability (spread)

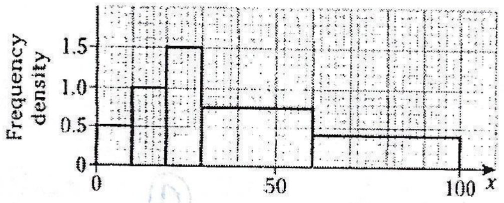
	Advantages	Disadvantages
Range	It is easy to calculate. It represents the complete spread of data.	It is affected by extreme values.
Interquartile range	It is not unduly influenced by extreme values. It can be used to investigate extreme values.	It depends only on particular values when the data are ranked.
Standard deviation	It is calculated using all the data and so represents every item. It is calculated using a mathematical formula, so calculators can be programmed to find it. It is very useful for further analysis. It is useful in comparing two sets of data, for example by showing which is more consistent.	It can be unduly affected by one or two extreme values. For a single set of data, its value is difficult to interpret.

Diagrammatic representation

	Advantages	**Disadvantages**
Bar chart	It shows the mode. Different sets of data can be compared using comparative bar charts.	It is only useful for qualitative data.
Pie chart	It shows the proportions of each quantity.	It has limited use with quantitative data. It does not show frequencies.
Vertical line diagram	It shows the mode clearly. It gives an idea of the shape of the distribution.	It is only useful for illustrating a small number of values.
Stem-and-leaf diagram	It shows all the original data. It shows the shape of the distribution. The mode, median and quartiles can be found from the diagram. It is useful for comparing two sets of data.	It is not suitable for large amounts of data.
Histogram	It can represent groups of different widths. It shows whether the distribution is symmetrical or skew. The mean and standard deviation can be estimated from the histogram.	The visual impact can be altered by choosing different groups. Two distributions cannot be shown on the same diagram.
Cumulative frequency graphs	The median and quartiles can be estimated from the graph. Sets of data can be compared by drawing graphs on the same diagram.	The visual impact can be altered by using different scales.
Box-and-whisker plot	It is easy to see whether the distribution is symmetrical or whether there is a tail to the left or right. It can be used to investigate extreme values (outliers). It is easy to see the range and interquartile range. You can compare two or more sets of data by drawing plots on the same diagram.	It does not show frequencies.

Histogram (grouped data)

Frequency density $= \dfrac{\text{frequency}}{\text{interval width}}$



Area of bar = frequency in that interval

Total area = total frequency

There are no gaps between the bars.

Interval width = upper class boundary − lower class boundary

The modal class is represented by the highest bar.

Mean and standard deviation

For raw data:

$$\bar{x} = \frac{\sum x}{n} \qquad \text{standard deviation} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

For data in a frequency table:

Use the calculation versions.

$$\bar{x} = \frac{\sum xf}{\sum f} \qquad \text{standard deviation} = \sqrt{\frac{\sum(x-\bar{x})^2 f}{\sum f}} = \sqrt{\frac{\sum x^2 f}{\sum f} - \bar{x}^2}$$

When data are **grouped**, use the **mid-interval value** to represent the interval, where

mid-interval value $= \frac{1}{2}$(l.c.b. + u.c.b.)

Combining sets of data for x and y:

$$\text{mean} = \frac{\sum x + \sum y}{n_1 + n_2} \qquad \text{standard deviation} = \sqrt{\frac{\sum x^2 + \sum y^2}{n_1 + n_2} - (\text{mean})^2}$$

Finding the mean and standard deviation using $\sum(x - a)$ and $\sum(x - a)^2$

Using formulae

You could use the following formulae:

$$\bar{x} = \underset{\uparrow \text{ mean of } (x-a)}{\frac{\sum(x-a)}{n}} + a \qquad \text{s.d. of } x = \sqrt{\frac{\sum(x-a)^2}{n} - \Bigg(\underset{\uparrow \text{ mean of } (x-a)}{\frac{\sum(x-a)}{n}}\Bigg)^2}$$

Median and quartiles

Median

For a set of n numbers arranged in ascending order:

- when n is odd, the median is the middle value
- when n is even, the median is the mean of the two middle values.

This is summarised by saying that the **median** is the $\frac{1}{2}(n+1)^{\text{th}}$ value.

Quartiles

- The **lower quartile**, Q_1, is the median of all the values **before** the median.
- The **upper quartile**, Q_3, is the median of all the values **after** the median.

List the values in order and count through the data values to find the median and quartiles.

When estimating the median and quartiles from a cumulative frequency graph with a large total frequency, the following are usually used:

median $= \frac{1}{2}n^{\text{th}}$ value, lower quartile $= \frac{1}{4}n^{\text{th}}$ value, upper quartile $= \frac{3}{4}n^{\text{th}}$ value.

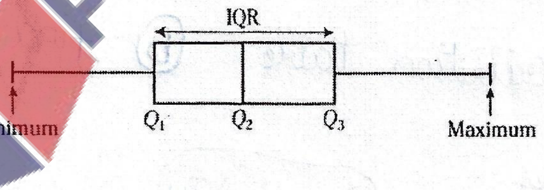
Ranges

Range = highest value − lowest value

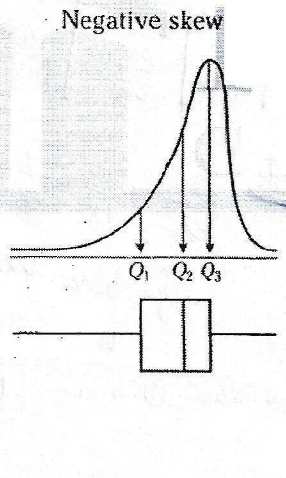
Interquartile range (IQR) = upper quartile − lower quartile = $Q_3 - Q_1$

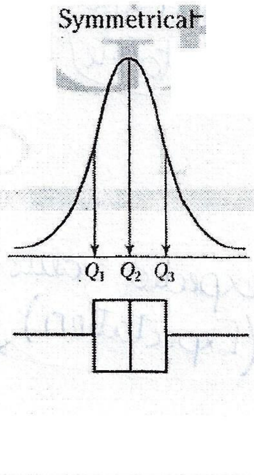
Box-and-whisker plot

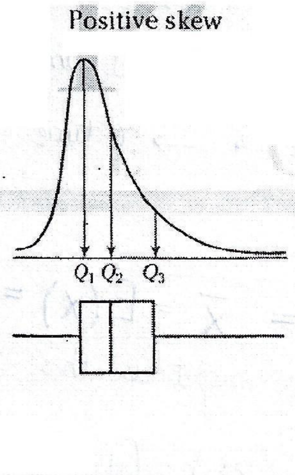
- The ends of the whiskers are at the minimum and maximum values.
- The ends of the box are drawn at the lower quartile Q_1 and the upper quartile Q_3.
- The line in the box is drawn at the median Q_2.



Shape of a distribution







Topic: Permutations and Combinations

ARRANGEMENTS IN A LINE

Arrangements of distinct items

The number of different arrangements of n distinct items is

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$$

n must be a positive integer

Example 1:

Each of the letters of the word **CAMBRIDGE** is written on a card and the cards are placed in a line.

(i) How many different arrangements are there?

(ii) How many arrangements begin with **CAM**?

There are 9 letters and they are distinct (no repeat letters).

(i) Number of arrangements $= 9! = 362\,880$

On calculator using factorials:

$\boxed{9}\,\boxed{x!}\,\boxed{=}$ 362 880

(ii) Place the first three letters C A M

There are now only 6 letters to be placed.

Number of arrangements $= 6! = 720$

$\boxed{6}\,\boxed{x!}\,\boxed{=}$ 720

Arrangements when items are not distinct

The number of different arrangements of n items of which p of one type are alike, q of another type are alike, r of another type are alike, and so on, is $\dfrac{n!}{p! \times q! \times r! \times \ldots}$.

Example 2:

Find the number of different arrangements using all ten letters of the word STATISTICS.

There are 10 letters to be arranged.

Consider the letters that are repeated.

T occurs 3 times.

S occurs 3 times.

I occurs 2 times.

So the number of different arrangements

$$= \frac{10!}{3! \times 3! \times 2!} = 50\,400$$

3 Ts　3 Ss　2 Is

Arrangements when there are restrictions

Example 3:

The word **ARGENTINA** includes the four consonants **R, G, N, T** and the three vowels **A, E, I.**

(i) Find the number of different arrangements using all nine letters.

(ii) How many of these arrangements have a consonant at the beginning, then a vowel, then another consonant, and so on alternately?

Cambridge Paper 6 Q1 N04

(i) There are 9 letters.
A occurs twice and N occurs twice.
So the number of different arrangements

$$= \frac{9!}{2!2!} = 90\,720$$

2 As   2 Ns

(ii) You want a consonant at the beginning (C), then a vowel (V), then another consonant (C), and so on

i.e. C V C V C V C V C

First place the consonants, noting that there are repeats.

Number of arrangements of the consonants R, G, N, N, T

$$= \frac{5!}{2!} = 60$$

2 Ns

Now place the vowels, noting that there are repeats.

The four vowels A, A, E, I can then slot into the four spaces between the consonants.

Number of arrangements of the vowels A, A, E, I

$$= \frac{4!}{2!} = 12$$

2 As

So the number of arrangements with consonants and vowels placed alternately

$$= \frac{5!}{2!} \times \frac{4!}{2!} = 60 \times 12 = 720$$

Arrangements when repetitions are allowed

Example 4:

How many 5-digit **odd** numbers can be made with the digits 2, 3, 6, 7, 8

(i) if repetitions are not allowed, for example, 63 287,

(ii) if repetitions are allowed, for example, 88 663?

Since the number is odd, it must end in a 3 or a 7.

(i) If repetitions are not allowed:

There are 2 choices for the last digit.

The remaining 4 digits can be arranged in 4! ways.

So, if repetitions are not allowed, there are $2 \times 4! = 48$ possible odd numbers.

(ii) If repetitions are allowed:

There are 5 choices for each of the first 4 digits, but only 2 choices for the last digit.

So, if repetitions are allowed, there are

$5 \times 5 \times 5 \times 5 \times 2 = 5^4 \times 2 = 1250$ possible odd numbers.

Example 5:

(i) Safebank requires its customers to use a four-digit PIN to access their account. Customers can choose any set of 4 digits from 0, 1, 2, ..., 9 and digits may be repeated. How many possible four-digit PINs are there?

(ii) Smartbank requires its customers to use a password consisting of four lower-case letters. Repetitions are allowed. How many possible passwords are there?

(iii) Excelbank requires its customers to use a pass-code consisting of four letters followed by four digits. Repetitions are allowed. How many possible pass-codes are there?

(i) There are 10 choices for each of the four digits, so number of possible PINs $= 10 \times 10 \times 10 \times 10 = 10^4 = 10\,000$

(ii) There are 26 choices for each letter, so number of possible passwords $= 26 \times 26 \times 26 \times 26 = 26^4 = 456\,976$

(iii) Number of possible pass-codes $= 26^4 \times 10^4 = 4\,569\,760\,000$

PERMUTATIONS OF r ITEMS FROM n ITEMS

The number of **permutations**, or **ordered arrangements**, of r items taken from n distinct items is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

${}_nP_r$ can also be written ${}^nP_r$

In permutations, the **order** of the selection **matters**.

Example 6:

Find how many numbers bigger than 30 000 but smaller than 40 000 can be formed from the digits 2, 3, 4, 5, 6, 7, 8 if no digit is repeated and the number must be a multiple of 5.

The number must have 5 digits.

It must start with 3 and end with 5, so fix these two digits

$$\boxed{3}\quad\boxed{*}\quad\boxed{*}\quad\boxed{*}\quad\boxed{5}$$

There are now three spaces to fill and the digits must be taken from the five digits 2, 4, 6, 7, 8.

Number of ways to fill the three remaining places

$$= {}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$$

Directly on calculator:

$\boxed{5}\boxed{{}_nP_r}\boxed{3}\boxed{=}$ 60

So 60 different numbers can be made which satisfy the conditions.

Example 7:

A security code consists of 4 letters chosen from A, B, C, D, E, F, G followed by 3 digits chosen from 0, 1, 2, 3, 4, 5.

Examples are BCDG102 (without repetitions) and CCDD225 (with repetitions).

Show that more than five times as many codes can be made when repetitions are allowed than when repetitions are not allowed.

There are 7 letters and 6 digits.

When repetitions are not allowed:

Number of arrangements of 4 letters from 7 letters

$$= {}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 840$$

Number of arrangements of 3 digits from 6 digits

$$= {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$$

Number of possible codes (repetitions not allowed)

$$= 840 \times 120 = 100\,800$$

When repetitions are allowed:

There are 7 choices for each of the 4 letters, so number of choices = 7^4.

There are 6 choices for each of the 3 digits, so number of choices = 6^3.

Number of possible codes (repetitions allowed)

$$= 7^4 \times 6^3$$

$$= 518\,616 > 5 \times 100\,800$$

So more than five times as many codes can be made when repetitions are allowed.

COMBINATIONS OF *r* ITEMS FROM *n* ITEMS

A **combination** is a selection of some items where the order of the selected items **does not matter**.

The number of **combinations** of r items chosen from n distinct items is given by

$$\underbrace{\binom{n}{r}}_{n\text{ choose }r} = {}_nC_r = \frac{n!}{r!(n-r)!}$$

${}_nC_r$ can also be written ${}^nC_r$

In **combinations**, the **order** of the selection **does not matter**.

Example 1:

Issam has 11 different CDs of which 6 are pop music, 3 are jazz and 2 are classical. Issam makes a selection of 2 pop music CDs, 2 jazz CDs and 1 classical CD.

How many different possible selections can be made?

Cambridge Paper 6 Q3(ii) part J08

Number of ways to choose 2 pop music CDs from 6 pop music CDs = $\binom{6}{2}$

Number of ways to choose 2 jazz CDs from 3 jazz CDs = $\binom{3}{2}$

Number of ways to choose 1 classical CD from 2 classical CDs = $\binom{2}{1}$

So number of possible selections

$$= \binom{6}{2} \times \binom{3}{2} \times \binom{2}{1} = 15 \times 3 \times 2 = 90$$

Example 2:

A collection of 18 books contains one Harry Potter book. Linda is going to choose 6 of these books to take on holiday.

(i) In how many ways can she choose 6 books?

(ii) How many of these choices will include the Harry Potter book?

Cambridge Paper 6 Q6(a) N03

(i) Number of ways to choose 6 books from 18 books

$$= \binom{18}{6} = \frac{18!}{6!12!} = 18\,564$$

(ii) If the Harry Potter book is included, Linda has to choose the other 5 books from 17 books.

Number of ways including the Harry Potter book

$$= \binom{17}{5} = \frac{17!}{5!12!} = 6\,188$$

Example 3:

A committee of 5 people is to be chosen from 6 men and 4 women. In how many ways can this be done:

(i) if there must be 3 men and 2 women on the committee,

(ii) if there must be more men than women on the committee,

(iii) if there must be 3 men and 2 women, and one particular woman refuses to be on the committee with one particular man?

Cambridge Paper 6 Q5 J03

(i) Number of ways to choose 3 men from 6 men = $\binom{6}{3}$

Number of ways to choose 2 women from 4 women = $\binom{4}{2}$

So, number of ways to choose committee

$$\begin{aligned} &= \binom{6}{3} \times \binom{4}{2} \\ &= 20 \times 6 \\ &= 120 \end{aligned}$$

(ii) If there are more men than women on the committee there could be

	Number of ways
5 men, 0 women	$\binom{6}{5} = 6$
4 men, 1 woman	$\binom{6}{4} \times \binom{4}{1} = 15 \times 4 = 60$
3 men, 2 women	$\binom{6}{3} \times \binom{4}{2} = 20 \times 6 = 120$

Total number of ways = 6 + 60 + 120 = 186.

(iii) Denoting the particular man by M and the particular woman by W, first consider the number of ways with both M and W on the committee.

You now need to choose two more men and one more woman.

Number of ways to choose 2 men from 5 men = $\binom{5}{2}$

Number of ways to choose 1 woman from 3 women = $\binom{3}{1}$

So number of ways with **both M and W** on the committee

$$= \binom{5}{2} \times \binom{3}{1} = 10 \times 3 = 30$$

Now subtract this from the number of ways of choosing the committee with no restrictions.

From part (i), total number of ways to choose committee = 120

So number of ways with **not both M and W** on the committee
$= 120 - 30$
$= 90$

Combination of r items from n items when the items are not distinct

Example 4:

Three letters are selected at random from the letters of the word BIOLOGY.

Find the total number of selections.

*The answer is **not** $\binom{7}{3}$ as you might expect.*

Because there are two letters O, you need to find the number of selections with

no letters O
one letter O
two letters O

and then add these together.

For example
B, L and Y

Number of selections with no letter O
= number of ways to choose **three** letters from B, I, L, G, Y
$= \binom{5}{3} = 10$

For example
O, B and L

Number of selections with one letter O
= number of ways to choose **two** letters from B, I, L, G, Y
$= \binom{5}{2} = 10$

For example
O, O and B

Number of selections with two letters O
= number of ways to choose **one** letter from B, I, L, G, Y
= 5

Therefore, total number of selections = 10 + 10 + 5 = 25

New Chapter

Summary

Arrangements in a line

The number of different arrangements of n distinct objects is

n must be a positive integer

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$$

The number of different arrangements of n items of which p are alike is $\frac{n!}{p!}$.

The number of different arrangements of n items of which p of one type are alike, q of another type are alike, r of another type are alike, and so on, is $\frac{n!}{p! \times q! \times r! \times \ldots}$.

By definition, $0! = 1$

Permutations and combinations

Permutations

The number of **permutations** of r items taken from n distinct items is

$${}_nP_r = \frac{n}{(n-r)!}$$

In permutations, **order matters.**

Combinations

The number of **combinations** of r items taken from n distinct items is

$${}_nC_r = \binom{n}{r} = \frac{n}{r!(n-r)!}$$

In combinations, **order does not matter.**

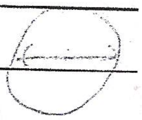
Area of hexagon = $\frac{3\sqrt{3}}{2} \times a^2$

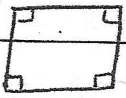
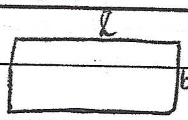
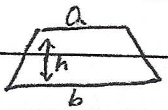
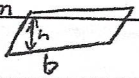
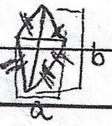
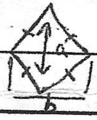
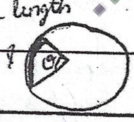
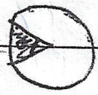
Date : 18/10/2017

Mensuration

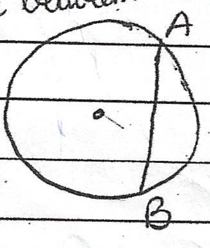
$1l = 1000 cm^3$

$C = 2\pi r$

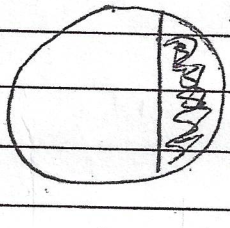


Shape	Area	Perimeter
Square 	$A = l^2$	$P = 4l$
Rectangle 	$A = l \times b$	$P = 2(l+b)$
Trapezium 	$A = \frac{1}{2} \times h(a+b)$	P = Sum of all sides
Parallelogram 	$A = b \times h$ (or) $A = ab\sin c$	P = Sum of all sides
kite 	$A = \frac{1}{2} \times a \times b$ (or) $\frac{1}{2} \times$ Product of diagonal	P = 2 x sum of non adjacent sides
Rhombus 	$A = \frac{1}{2} \times a \times b$	$P = 4 \times l$
Circle 	$A = \pi r^2$	circumference of circle $= 2\pi r$ (or) πd
Semi-circle 	$A = \frac{1}{2} \pi r^2$	$C = \pi r + 2r$
Arc length 	Formula of length $l = \frac{\theta}{360} \times 2\pi r$	
Sector 	$A = \frac{\theta}{360} \times \pi r^2$	$P = 2r + l$
Equilateral 	Area $= \frac{\sqrt{3}}{4} a^2$	

line between A and B is called Chord



Shaded is called Segment



$\Delta = A = \frac{1}{2} \times ab \sin C$

$\Delta = A = \frac{1}{2} \times a \times h$

Reflection

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection in 'X' axis

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection in 'Y' axis

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ = y=x reflection

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ = Reflection y = -x.

Rotation

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ +90° anti (0,0)

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ − 90° clock "

"

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ± 180°. "

$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ k = S.F Center (0,0)

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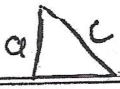
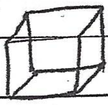
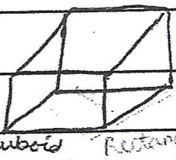
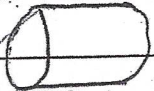
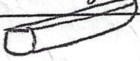
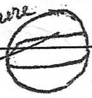
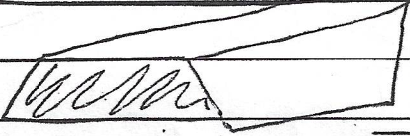
Date :



$a = \sqrt{c^2 - b^2}$ $b = \sqrt{c^2 - a^2}$

$c = \sqrt{a^2 + b^2}$

Shape	Volume	C.S.A	T.S.A
cube 	$V = l^3$	~~$4l^2$~~	$6l^2$
 cuboid Rectangular	$V = l \times b \times h$	~~$2h(l+b)$~~	$2(lb + bh + lh)$
 cylinder	$V = \pi r^2 h$	$2\pi r h$	$2\pi r(r+h)$ $2\pi r h + 2\pi r^2$
Hollow-cylinder 	$V = \frac{1}{2}\pi r^2 h$	$\pi r h$	$\frac{1}{2}\pi r(r+h)$
Sphere 	$V = \frac{4}{3}\pi r^3$	$4\pi r^2$	$4\pi r^2$
Hemi-sphere 	$V = \frac{2}{3}\pi r^3$	$2\pi r^2$	$3\pi r^2$
Cone 	$V = \frac{1}{3}\pi r^2 h$	$\pi r l$	$\pi r(r + l)$ $\pi r l + \pi r^2$
Pyramid 	$V = \frac{1}{3}(\text{base area}) \times h$		Sum of area of sides and base



prism

$V \cdot \text{rate} = \underset{cm/s}{\text{rate}} \times \text{Area}$

$V = (\text{Area of cross-section}) \times \text{length}$

$\underset{cm^3}{\text{Flow of water}} = \frac{1}{4} \times \pi (\text{Pipe diameter})^2 \times \underset{cm.}{\text{Velocity}}$

Time = Distance

Magnitude of a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ $\sqrt{x^2+y^2}$

Polygons: exterior angle add up to 360.

Sum of interior angle = $(n-2) \times 180°$

3 = triangle 4 = quadrilateral
6 = hexagon 5 = pentagon.
7 = heptagon 8 = octagon
9 = nonagon 10 = decagon.

$A' = \frac{1}{ad-bc} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

C.W → −
anti C.W → +

Reflection = mirror line

Rotation = center, angle, direction.

Translation = vector $\begin{bmatrix} x \\ y \end{bmatrix}$

Enlargement : S.F, center, S.F = $\frac{\text{length of I}}{\text{length of O}}$

Area of image = $(S.F)^2$ area of object

~~A~~ (V of image = $(S.F)^3$ volume of object)

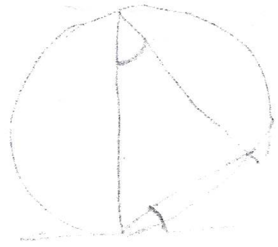
check again not sure

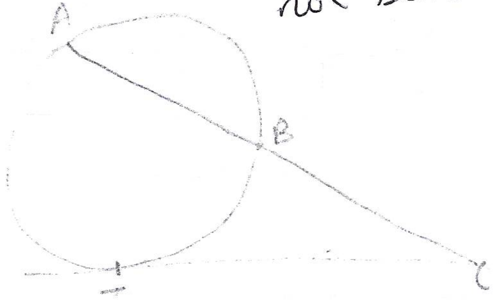
Repeated transformation.

$(\sqrt[n]{a})^m = a^{\frac{m}{n}}$.

$a^0 = 1$.

$a^{-n} = \frac{1}{a^n}$





$TC^2 = AC \times BC$

$$\begin{tikzcd}
m^3 \arrow[r, bend left, "\times 1000{,}000"] & cm^3 \arrow[r, bend left, "\times 1000"] \arrow[l, bend left, "\div 1000{,}000"] & mm^3 \arrow[l, bend left, "\div 1000"]
\end{tikzcd}$$

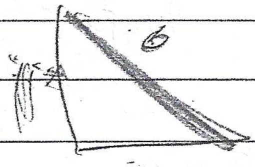
$(r \times c)$

$1l = 1000cm^3$

$1ml = 1cm^3$

$1kl = 1m^3$.

Formula's of Chapter 6 (Trigonometry)



1- Find length if 2 sides are avalible

~~$a^2 = b^2 + c^2$~~

$c^2 = b^2 + a^2$ b^2

2. To find side or angle (right angle - Δ)

SOH CAH TOA

$\underbrace{\text{SOH CAH TOA}}_{\text{sleep catto}}$

3. Sine rule :- (non right angle Δ) 2 len 2 ang

For side

$$\frac{A}{\sin A} = \frac{B}{\sin B} = \frac{C}{\sin C}$$

For angle

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}$$

4. Cosine rule :- ~~3 length 1 angle~~

To find length of side (3 sides, 1 angle)

- $a^2 = b^2 + c^2 - 2bc \cos A$
- $b^2 = a^2 + c^2 - 2ac \cos B$
- $c^2 = b^2 + a^2 - 2ab \cos C$

To find ~~length~~ angle of when (all sides are given)

- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
- $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Area of Δ non right angle

$= \dfrac{1}{2} \times a \times b \times \sin C$

right angle Δ

$= \dfrac{1}{2} \times b \times h$

C.f = add all before.

A.p → $a+(n-1)d$.

G.p → ar^{n-1}.

$$M.P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$f.d = \frac{f}{c.w}$$

if 90° to each other

$$g_1 \times g_2 = \boxed{-1}$$

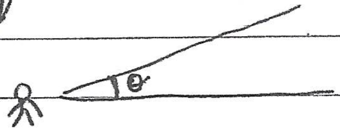
$y = mx + c$

$$\text{length} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Angle of Elevation

$a^b = c$

$b = \log_{(a)}(C)$



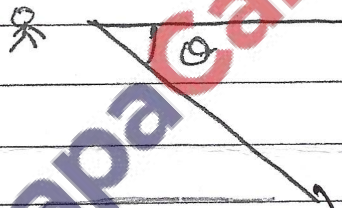
angle between normal and above

$2a$
$\uparrow$
$3a+b$
$\wedge\ \wedge$
$a+b+c$

an^2+bn+c

Angle of depression

$$S.I = \frac{Prt}{100}$$
(pure)



angle between normal and down.

$$C.I = P\left(1 \pm \frac{r}{100}\right)^{x}$$
(impure)

rate, year, price

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- congruent △ if same.

$a(b+c) = ab + ac$

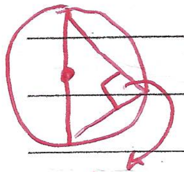
$(a+b)^2 = a^2+2ab+b^2$.

$(a-b)^2 = a^2-2ab+b^2$.

$a^2+b^2 = (a+b)^2-2ab$.

$a^2-b^2 = (a+b)(a-b)$.

- similar if angle are similar and ratio.



angle in semicircle.

acute → θ less than 90°.

obtuise → 90° < θ < 180°.

reflex → 180° < θ < 360°.

imp (completing square) :-

$$x^2 - ax + b = (x-p)^2 - \underline{q}$$

$$\pm\left(\frac{a}{2}\right)^2$$

$$x^2 - ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b \rightarrow \left(x-\frac{a}{2}\right)^2 - \left(\frac{a^2}{4}+b\right)$$

$= q$