4 The polynomial $x^4 - 2x^3 - 2x^2 + a$ is denoted by $f(x)$. It is given that $f(x)$ is divisible by $x^2 - 4x + 4$.

**(i)** Find the value of $a$. [3]

**(ii)** When $a$ has this value, show that $f(x)$ is never negative. [4]

9 Relative to the origin $O$, the point $A$ has position vector given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The line $l$ has equation $\mathbf{r} = 9\mathbf{i} - \mathbf{j} + 8\mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

**(i)** Find the position vector of the foot of the perpendicular from $A$ to $l$. Hence find the position vector of the reflection of $A$ in $l$. [5]

.................................................................................................................................................

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| | |
|---|---|
| 9(i) | *EITHER*: Find $\overline{AP}$ for a general point $P$ on $l$ with parameter $\lambda$, e.g.$(8 + 3\lambda, -3 - \lambda, 4 + 2\lambda)$ |
| | Equate scalar product of $\overline{AP}$ and direction vector of $l$ to zero and solve for $\lambda$ |
| | Obtain $\lambda = -\frac{5}{2}$ and foot of perpendicular $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 3\mathbf{k}$ |
| | Carry out a complete method for finding the position vector of the reflection of $A$ in $l$ |
| | Obtain answer $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ |

**10** The points $A$ and $B$ have position vectors, relative to the origin $O$, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \qquad \text{and} \qquad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line $l$ has vector equation

$$\mathbf{r} = (1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}.$$

**(i)** Show that $l$ does not intersect the line passing through $A$ and $B$. [4]

**(ii)** The point $P$ lies on $l$ and is such that angle $PAB$ is equal to 60°. Given that the position vector of $P$ is $(1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of $P$. [6]

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| | | | |
|---|---|---|---|
| **10 (i)** | State a vector equation for the line through $A$ and $B$, e.g. $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + s(\mathbf{i} - \mathbf{j})$ | B1 | |
| | Equate at least two pairs of components of general points on $AB$ and $l$, and solve for $s$ or for $t$ | M1 | |
| | Obtain correct answer for $s$ or $t$, e.g. $s = -6, 2, -2$ when $t = 3, -1, -1$ respectively | A1 | |
| | Verify that all three component equations are not satisfied | A1 | [4] |
| **(ii)** | State or imply a direction vector for $AP$ has components $(-2t, 3 + t, -1-t)$, or equivalent | B1 | |
| | State or imply cos 60° equals $\dfrac{\overrightarrow{AP}.\overrightarrow{AB}}{\lvert\overrightarrow{AP}\rvert.\lvert\overrightarrow{AB}\rvert}$ | M1* | |
| | Carry out correct processes for expanding the scalar product and expressing the product of the moduli in terms of $t$, in order to obtain an equation in $t$ in any form | M1(dep*) | |
| | Obtain the given equation $3t^2 + 7t + 2 = 0$ correctly | A1 | |
| | Solve the quadratic and use a root to find a position vector for $P$ | M1 | |
| | Obtain position vector $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ from $t = -2$, having rejected the root $t = -\frac{1}{3}$ for a valid reason | A1 | [6] |

7 With respect to the origin $O$, the position vectors of two points $A$ and $B$ are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point $P$ lies on the line through $A$ and $B$, and $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

**(i)** Show that $\overrightarrow{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$. [2]

**(ii)** By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of $\lambda$, find the value of $\lambda$ for which $OP$ bisects the angle $AOB$. [5]

**(iii)** When $\lambda$ has this value, verify that $AP : PB = OA : OB$. [1]

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**7 (i)** Use a correct method to express $\overrightarrow{OP}$ in terms of $\lambda$ — M1

Obtain the given answer — A1 [2]

**(ii)** *EITHER*: Use correct method to express scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OP}$, or $\overrightarrow{OB}$ and $\overrightarrow{OP}$ in terms of $\lambda$ — M1

Using the correct method for the moduli, divide scalar products by products of moduli and express $\cos AOP = \cos BOP$ in terms of $\lambda$, or in terms of $\lambda$ and $OP$ — M1*

*OR1*: Use correct method to express $OA^2 + OP^2 - AP^2$, or $OB^2 + OP^2 - BP^2$ in terms of $\lambda$ — M1

Using the correct method for the moduli, divide each expression by twice the product of the relevant moduli and express $\cos AOP = \cos BOP$ in terms of $\lambda$, or $\lambda$ and $OP$ — M1*

Obtain a correct equation in any form, e.g. $\dfrac{9+2\lambda}{3\sqrt{(9+4\lambda+12\lambda^2)}} = \dfrac{11+14\lambda}{5\sqrt{(9+4\lambda+12\lambda^2)}}$ — A1

Solve for $\lambda$ — M1(dep*)

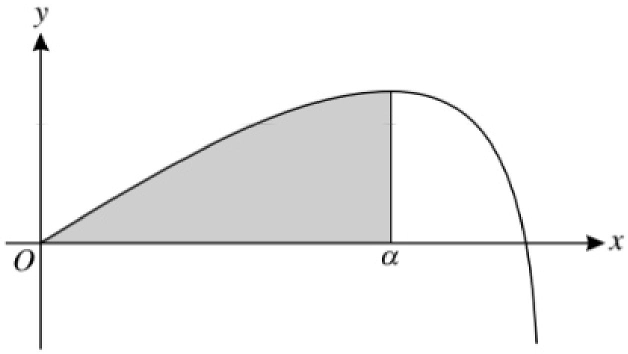
Obtain $\lambda = \frac{3}{8}$ — A1 [5]

[SR: The M1* can also be earned by equating $\cos AOP$ or $\cos BOP$ to a sound attempt at $\cos \frac{1}{2} AOB$ and obtaining an equation in $\lambda$. The exact value of the cosine is $\sqrt{(13/15)}$, but accept non-exact working giving a value of $\lambda$ which rounds to 0.375, provided the spurious negative root of the quadratic in $\lambda$ is rejected.]

[SR: Allow a solution reaching $\lambda = \frac{3}{8}$ after cancelling identical incorrect expressions for $OP$ to score 4/5. The marking will run M1M1A0M1A1, or M1M1A1M1A0 in such cases.]

**(iii)** Verify the given statement correctly — B1 [1]

5




The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The $x$-coordinate of the maximum point is $\alpha$ and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

**(i)** Show that $\alpha = \frac{2}{3}\pi$. [3]

**(ii)** Find the exact value of the area of the shaded region. [4]

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| | | | | |
|---|---|---|---|---|
| 5 | (i) | Differentiate to obtain $4\cos\frac{1}{2}x - \frac{1}{2}\sec^2\frac{1}{2}x$ | B1 | |
| | | Equate to zero and find value of $\cos\frac{1}{2}x$ | M1 | |
| | | Obtain $\cos\frac{1}{2}x = \frac{1}{2}$ and confirm $\alpha = \frac{2}{3}\pi$ | A1 | [3] |
| | (ii) | Integrate to obtain $-16\cos\frac{1}{2}x \ldots$ | B1 | |
| | | $\ldots + 2\ln\cos\frac{1}{2}x$ or equivalent | B1 | |
| | | Using limits 0 and $\frac{2}{3}\pi$ in $a\cos\frac{1}{2}x + b\ln\cos\frac{1}{2}x$ | M1 | |
| | | Obtain $8 + 2\ln\frac{1}{2}$ or exact equivalent | A1 | [4] |

**10** **(a)** The complex numbers $u$ and $w$ satisfy the equations

$$u - w = 4\mathrm{i} \quad \text{and} \quad uw = 5.$$

Solve the equations for $u$ and $w$, giving all answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real. [5]

**(b)** **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 + 2\mathrm{i}| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of $z$. [5]

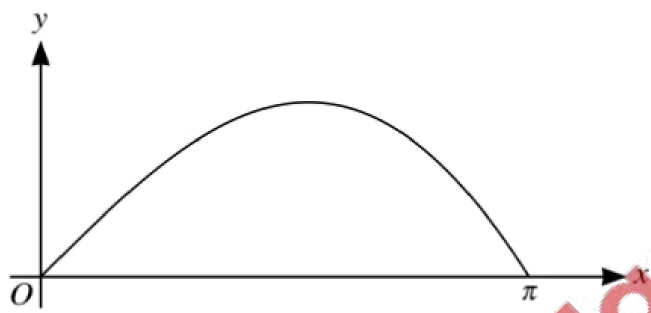
**(ii)** Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

9709_s12_ms_33.pdf



| | | | |
|---|---|---|---|
| **10 (a)** | EITHER: Eliminate $u$ or $w$ and obtain an equation in $w$ or in $u$ | M1 | |
| | Obtain a quadratic in $u$ or $w$, e.g. $u^2 - 4\mathrm{i}u - 5 = 0$ or $w^2 + 4\mathrm{i}w - 5 = 0$ | A1 | |
| | Solve a 3-term quadratic for $u$ or for $w$ | M1 | |
| | OR1: Having squared the first equation, eliminate $u$ or $w$ and obtain an equation in $w$ or $u$ | M1 | |
| | Obtain a 2-term quadratic in $u$ or $w$, e.g. $u^2 = -3 + 4\mathrm{i}$ | A1 | |
| | Solve a 2-term quadratic for $u$ or for $w$ | M1 | |
| | OR2: Using $u = a + \mathrm{i}b$, $w = c + \mathrm{i}d$, equate real and imaginary parts and obtain 4 equations in $a$, $b$, $c$ and $d$ | M1 | |
| | Obtain 4 correct equations | A1 | |
| | Solve for $a$ and $b$, or for $c$ and $d$ | M1 | |
| | Obtain answer $u = 1 + 2\mathrm{i}$, $w = 1 - 2\mathrm{i}$ | A1 | |
| | Obtain answer $u = -1 + 2\mathrm{i}$, $w = -1 - 2\mathrm{i}$ and no other | A1 | [5] |
| **(b) (i)** | Show point representing $2 - 2\mathrm{i}$ in relatively correct position | B1 | |
| | Show a circle with centre $2 - 2\mathrm{i}$ and radius 2 | B1√ | |
| | Show line for $\arg z = -\dfrac{1}{4}\pi$ | B1 | |
| | Show line for $\operatorname{Re} z = 1$ | B1 | |
| | Shade the relevant region | B1 | [5] |
| **(ii)** | State answer $2 + \sqrt{2}$, or equivalent (accept 3.41) | B1 | [1] |

**8**




The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

**(i)** Find $\frac{dy}{dx}$ and show that $4\frac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

**(ii)** Find the exact value of the area of the region enclosed by this part of the curve and the $x$-axis. [5]



## 9709_s14_ms_32.pdf

| | | | | |
|---|---|---|---|---|
| 8 (i) | Use product rule | M1 | |
| | Obtain derivative in any correct form | A1 | |
| | Differentiate first derivative using the product rule | M1 | |
| | Obtain second derivative in any correct form, e.g. $-\frac{1}{2}\sin\frac{1}{2}x - \frac{1}{4}x\cos\frac{1}{2}x - \frac{1}{2}\sin\frac{1}{2}x$ | A1 | |
| | Verify the given statement | A1 | 5 |
| (ii) | Integrate and reach $kx\sin\frac{1}{2}x + l\int\sin\frac{1}{2}x\,dx$ | M1* | |
| | Obtain $2x\sin\frac{1}{2}x - 2\int\sin\frac{1}{2}x\,dx$, or equivalent | A1 | |
| | Obtain indefinite integral $2x\sin\frac{1}{2}x + 4\cos\frac{1}{2}x$ | A1 | |
| | Use correct limits $x = 0$, $x = \pi$ correctly | M1(dep*) | |
| | Obtain answer $2\pi - 4$, or exact equivalent | A1 | 5 |

7 **(a)** It is given that $-1 + (\sqrt{5})\mathrm{i}$ is a root of the equation $z^3 + 2z + a = 0$, where $a$ is real. Showing your working, find the value of $a$, and write down the other complex root of this equation. [4]

**(b)** The complex number $w$ has modulus 1 and argument $2\theta$ radians. Show that $\dfrac{w-1}{w+1} = \mathrm{i}\tan\theta$. [4]



| | | | | | |
|---|---|---|---|---|---|
| 7 | (a) | *EITHER*: | Substitute and expand $(-1+\sqrt{5}\,\mathrm{i})^3$ completely | M1 | |
| | | | Use $\mathrm{i}^2 = -1$ correctly at least once | M1 | |
| | | | Obtain $a = -12$ | A1 | |
| | | | State that the other complex root is $-1-\sqrt{5}\,\mathrm{i}$ | B1 | |
| | | *OR1*: | State that the other complex root is $-1-\sqrt{5}\,\mathrm{i}$ | B1 | |
| | | | State the quadratic factor $z^2+2z+6$ | B1 | |
| | | | Divide the cubic by a 3-term quadratic, equate remainder to zero and solve for $a$ or, using a 3-term quadratic, factorise the cubic and determine $a$ | M1 | |
| | | | Obtain $a = -12$ | A1 | |
| | | *OR2:* | State that the other complex root is $-1-\sqrt{5}\mathrm{i}$ | B1 | |
| | | | State or show the third root is 2 | B1 | |
| | | | Use a valid method to determine $a$ | M1 | |
| | | | Obtain $a = -12$ | A1 | |
| | | *OR3:* | Substitute and use De Moivre to cube $\sqrt{6}\,\mathrm{cis}(114.1^\circ)$, or equivalent | M1 | |
| | | | Find the real and imaginary parts of the expression | M1 | |
| | | | Obtain $a = -12$ | A1 | |
| | | | State that the other complex root is $-1-\sqrt{5}\mathrm{i}$ | B1 | 4 |

**5** **(i)** The polynomial $\mathrm{f}(x)$ is of the form $(x-2)^2\mathrm{g}(x)$, where $\mathrm{g}(x)$ is another polynomial. Show that $(x-2)$ is a factor of $\mathrm{f}'(x)$. [2]

**(ii)** The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where $a$ and $b$ are constants, has a factor $(x-2)^2$. Using the factor theorem and the result of part **(i)**, or otherwise, find the values of $a$ and $b$. [5]



| | | | | | |
|---|---|---|---|---|---|
| **5** | (i) | Differentiate f(x) and obtain $\mathrm{f}'(x) = (x-2)^2\,\mathrm{g}'(x) + 2(x-2)\mathrm{g}(x)$ | | B1 | |
| | | Conclude that $(x-2)$ is a factor of $\mathrm{f}'(x)$ | | B1 | **2** |
| | (ii) | *EITHER:* | Substitute $x = 2$, equate to zero and state a correct equation, e.g. $32 + 16a + 24 + 4b + a = 0$ | B1 | |
| | | | Differentiate polynomial, substitute $x = 2$ and equate to zero or divide by $(x-2)$ and equate constant remainder to zero | M1* | |
| | | | Obtain a correct equation, e.g. $80 + 32a + 36 + 4b = 0$ | A1 | |
| | | *OR1:* | Identify given polynomial with $(x-2)^2(x^3 + Ax^2 + Bx + C)$ and obtain an equation in $a$ and/or $b$ | M1* | |
| | | | Obtain a correct equation, e.g. $\frac{1}{4}a - 4(4+a) + 4 = 3$ | A1 | |
| | | | Obtain a second correct equation, e.g. $-\frac{3}{4}a + 4(4+a) = b$ | A1 | |
| | | *OR2:* | Divide given polynomial by $(x-2)^2$ and obtain an equation in $a$ and $b$ | M1* | |
| | | | Obtain a correct equation, e.g. $29 + 8a + b + 0$ | A1 | |
| | | | Obtain a second correct equation, e.g. $176 + 47a + 4b = 0$ | A1 | |
| | | Solve for $a$ or for $b$ | | M1(dep*) | |
| | | Obtain $a = -4$ and $b = 3$ | | A1 | **5** |

**1** Find the set of values of $x$ satisfying the inequality

$$|x + 2a| > 3|x - a|,$$

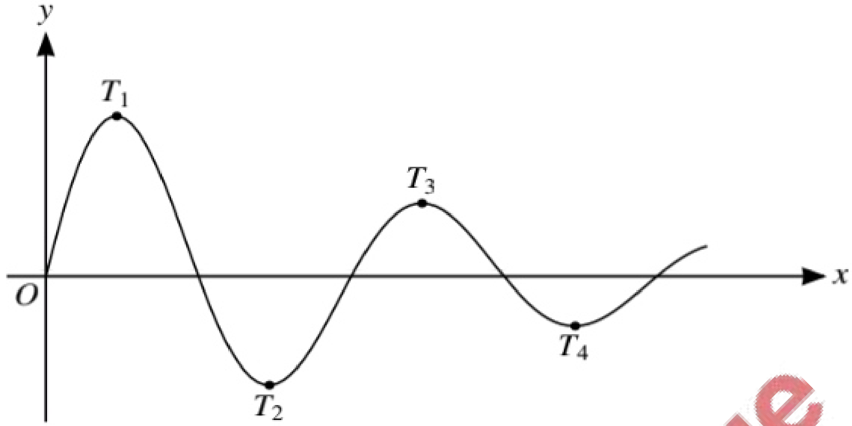
where $a$ is a positive constant. [4]

## 9709_s14_ms_32.pdf



| | | | | |
|---|---|---|---|---|
| **1** | *EITHER:* | State or imply non-modular inequality $(x+2a)^2 > (3(x-a))^2$, or corresponding quadratic equation, or pair of linear equations $(x+2a) = \pm 3(x-a)$ | B1 | |
| | | Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations for $x$ | M1 | |
| | | Obtain critical values $x = \frac{1}{4}a$ and $x = \frac{5}{2}a$ | A1 | |
| | | State answer $\frac{1}{4}a < x < \frac{5}{2}a$ | A1 | |
| | *OR:* | Obtain critical value $x = \frac{5}{2}a$ from a graphical method, or by inspection, or by solving a linear equation or inequality | B1 | |
| | | Obtain critical value $x = \frac{1}{4}a$ similarly | B2 | |
| | | State answer $\frac{1}{4}a < x < \frac{5}{2}a$ | B1 | **4** |
| | | [Do not condone $\leqslant$ for $<$.] | | |

**10**




The diagram shows the curve $y = 10e^{-\frac{1}{2}x}\sin 4x$ for $x \geqslant 0$. The stationary points are labelled $T_1$, $T_2$, $T_3$, … as shown.

**(i)** Find the $x$-coordinates of $T_1$ and $T_2$, giving each $x$-coordinate correct to 3 decimal places. [6]

**(ii)** It is given that the $x$-coordinate of $T_n$ is greater than 25. Find the least possible value of $n$. [4]

## 9709_s14_ms_31.pdf

| | | | | |
|---|---|---|---|---|
| **10** | **(i)** | Use of product or quotient rule | M1 | |
| | | Obtain $-5e^{-\frac{1}{2}x}\sin 4x + 40e^{-\frac{1}{2}x}\cos 4x$ | A1 | |
| | | Equate $\frac{dy}{dx}$ to zero and obtain $\tan 4z = k$ or $R\cos(4x \pm \alpha)$ | M1 | |
| | | Obtain $\tan 4x = 8$ or $\sqrt{65}\cos\left(4x \pm \tan^{-1}\frac{1}{8}\right)$ | A1 | |
| | | Obtain 0.362 or 20.7° | A1 | |
| | | Obtain 1.147 or 65.7° | A1 | [6] |
| | **(ii)** | State or imply that $x$-coordinates of $T_n$ are increasing by $\frac{1}{4}\pi$ or 45° | B1 | |
| | | Attempt solution of inequality (or equation) of form $x_1 + (n-1)k\pi\ .\ 25$ | M1 | |
| | | Obtain $n > \frac{4}{\pi}(25 - 0.362) + 1$, following through on their value of $x_1$ | A1√ | |
| | | n = 33 | A1 | [4] |

**2** Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\,dx.$$ [5]

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| | | | |
|---|---|---|---|
| **2** | State $\frac{du}{dx} = 3\sec^2 x$ or equivalent | B1 | |
| | Express integral in terms of $u$ and $du$ (accept unsimplified and without limits) | M1 | |
| | Obtain $\int \frac{1}{3} u^{\frac{1}{2}}\,du$ | A1 | |
| | Integrate $Cu^{\frac{1}{2}}$ to obtain $\frac{2C}{3} u^{\frac{3}{2}}$ | M1 | |
| | Obtain $\frac{14}{9}$ | A1 | [5] |

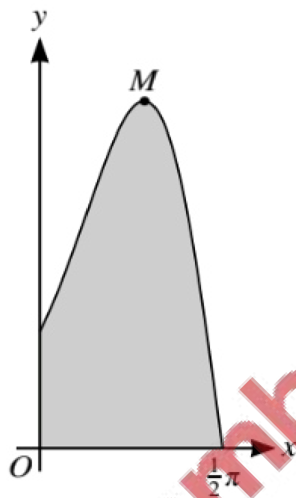
1 (i) Simplify $\sin 2\alpha \sec \alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos \beta = 0$, find the exact value of $\cos \beta$. [3]

## 9709_s14_ms_31.pdf

| | | | | |
|---|---|---|---|---|
| 1 | (i) | State $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\sec\alpha = 1/\cos\alpha$ | B1 | |
| | | Obtain $2\sin\alpha$ | B1 | [2] |
| | (ii) | Use $\cos 2\beta = 2\cos^2\beta - 1$ or equivalent to produce correct equation in $\cos\beta$ | B1 | |
| | | Solve three-term quadratic equation for $\cos\beta$ | M1 | |
| | | Obtain $\cos\beta = \frac{1}{3}$ only | A1 | [3] |

**9**




The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point $M$.

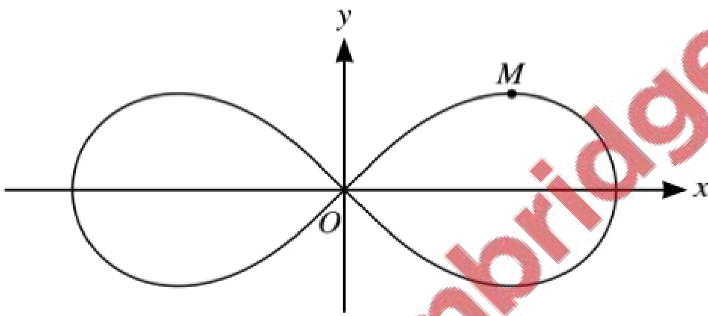
**(i)** Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. [5]

**(ii)** Find the $x$-coordinate of $M$, giving your answer correct to 3 decimal places. [6]



| | | | | |
|---|---|---|---|---|
| **9** | **(i)** | Substitute for $x$ and d$x$ throughout using $u = \sin x$ and d$u = \cos x$ d$x$, or equivalent | M1 | |
| | | Obtain integrand $e^{2u}$ | A1 | |
| | | Obtain indefinite integral $\frac{1}{2}e^{2u}$ | A1 | |
| | | Use limits $u = 0$, $u = 1$ correctly, or equivalent | M1 | |
| | | Obtain answer $\frac{1}{2}(e^2 - 1)$, or exact equivalent | A1 | **5** |

6




The diagram shows the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and one of its maximum points $M$. Find the coordinates of $M$. [7]

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| | | | |
|---|---|---|---|
| 6 | Obtain correct derivative of RHS in any form | B1 | |
| | Obtain correct derivative of LHS in any form | B1 | |
| | Set $\frac{dy}{dx}$ equal to zero and obtain a horizontal equation | M1 | |
| | Obtain a correct equation, e.g. $x^2 + y^2 = 1$, from correct work | A1 | |
| | By substitution in the curve equation, or otherwise, obtain an equation in $x^2$ or $y^2$ | M1 | |
| | Obtain $x = \frac{1}{2}\sqrt{3}$ | A1 | |
| | Obtain $y = \frac{1}{2}$ | A1 | 7 |

8 **(i)** By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

**(ii)** Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

**(iii)** Hence solve the equation

$$x^3 - x + \tfrac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]



| | | | |
|---|---|---|---|
| **8 (i)** | Use $\sin(A + B)$ formula to express $\sin 3\theta$ in terms of trig. functions of $2\theta$ and $\theta$ | M1 | |
| | Use correct double angle formulae and Pythagoras to express $\sin 3\theta$ in terms of $\sin\theta$ | M1 | |
| | Obtain a correct expression in terms of $\sin\theta$ in any form | A1 | |
| | Obtain the given identity | A1 | **[4]** |
| | [SR: Give M1 for using correct formulae to express RHS in terms of $\sin\theta$ and $\cos 2\theta$, then M1A1 for expressing in terms of $\sin\theta$ and $\sin 3\theta$ only, or in terms of $\cos\theta$, $\sin\theta$, $\cos 2\theta$ and $\sin 2\theta$, then A1 for obtaining the given identity.] | | |







| | | | |
|---|---|---|---|
| **(ii)** | Substitute for $x$ and obtain the given answer | B1 | **[1]** |
| **(iii)** | Carry out a correct method to find a value of $x$ | M1 | |
| | Obtain answers 0.322, 0.799, –1.12 | A1 + A1 + A1 | **[4]** |
| | [Solutions with more than 3 answers can only earn a maximum of A1 + A1.] | | |

**7** In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is $R$ million dollars when the rate of tax is $x$ dollars per litre. The variation of $R$ with $x$ is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where $R$ and $x$ are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

**(i)** Solve the differential equation and obtain an expression for $R$ in terms of $x$. [6]

**(ii)** This model predicts that $R$ cannot exceed a certain amount. Find this maximum value of $R$. [3]



| | | | | |
|---|---|---|---|---|
| 7 | (i) | Separate variables correctly and attempt to integrate at least one side | B1 | |
| | | Obtain term $\ln R$ | B1 | |
| | | Obtain $\ln x - 0.57x$ | B1 | |
| | | Evaluate a constant or use limits $x = 0.5$, $R = 16.8$, in a solution containing terms of the form $a\ln R$ and $b\ln x$ | M1 | |
| | | Obtain correct solution in any form | A1 | |
| | | Obtain a correct expression for $R$, e.g. $R = xe^{(3.80 - 0.57x)}$, $R = 44.7xe^{-0.57x}$ or $R = 33.6xe^{(0.285 - 0.57x)}$ | A1 | **[6]** |
| | (ii) | Equate $\frac{dR}{dx}$ to zero and solve for $x$ | M1 | |
| | | State or imply $x = 0.57^{-1}$, or equivalent, e.g. 1.75 | A1 | |
| | | Obtain $R = 28.8$ (allow 28.9) | A1 | **[3]** |

6 It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes $e^x$. [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]



| | | | | |
|---|---|---|---|---|
| 6 | (i) | Integrate and reach $bx\ln 2x - c\int x.\frac{1}{x}\,dx$, or equivalent | M1* | |
| | | Obtain $x\ln 2x - \int x.\frac{1}{x}\,dx$, or equivalent | A1 | |
| | | Obtain integral $x\ln 2x - x$, or equivalent | A1 | |
| | | Substitute limits correctly and equate to 1, having integrated twice | M1(dep*) | |
| | | Obtain a correct equation in any form, e.g. $a\ln 2a - a + 1 - \ln 2 = 1$ | A1 | |
| | | Obtain the given answer | A1 | [6] |
| | (ii) | Use the iterative formula correctly at least once | M1 | |
| | | Obtain final answer 1.94 | A1 | |
| | | Show sufficient iterations to 4 d.p. to justify 1.94 to 2d.p. or show that there is a sign change in the interval (1.935, 1.945). | A1 | [3] |

**10** By first using the substitution $u = e^x$, show that

$$\int_0^{\ln 4} \frac{e^{2x}}{e^{2x} + 3e^x + 2}\, dx = \ln\left(\tfrac{8}{5}\right).$$ [10]

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| | | | |
|---|---|---|---|
| **10** | State or imply $\frac{du}{dx} = e^x$ | B1 | |
| | Substitute throughout for $x$ and d$x$ | M1 | |
| | Obtain $\int \frac{u}{u^2 + 3u + 2}\, du$ or equivalent (ignoring limits so far) | A1 | |
| | State or imply partial fractions of form $\frac{A}{u+2} + \frac{B}{u+1}$, following their integrand | B1 | |
| | Carry out a correct process to find at least one constant for their integrand | M1 | |
| | Obtain correct $\frac{2}{u+2} - \frac{1}{u+1}$ | A1 | |
| | Integrate to obtain $a\ln(u+2) + b\ln(u+1)$ | M1 | |
| | Obtain $2\ln(u+2) - \ln(u+1)$ or equivalent, follow their $A$ and $B$ | A1√ | |
| | Apply appropriate limits and use at least one logarithm property correctly | M1 | |
| | Obtain given answer $\ln\frac{8}{5}$ legitimately | A1 | [10] |
| | **SR** for integrand $\frac{u^2}{u(u+1)(u+2)}$ | | |
| | State or imply partial fractions of form $\frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+2}$ | (B1) | |
| | Carry out a correct process to find at least one constant | (M1) | |
| | Obtain correct $\frac{2}{u+2} - \frac{1}{u+1}$ | (A1) | |
| | …complete as above. | | |

8 The variables $x$ and $y$ are related by the differential equation

$$\frac{dy}{dx} = \tfrac{1}{5}xy^{\frac{1}{2}}\sin\left(\tfrac{1}{3}x\right).$$

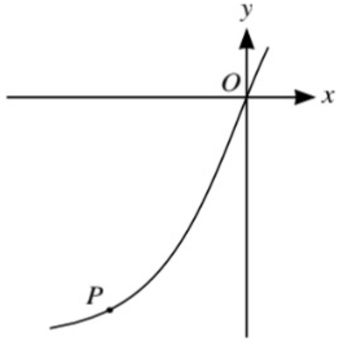
**(i)** Find the general solution, giving $y$ in terms of $x$. [6]

**(ii)** Given that $y = 100$ when $x = 0$, find the value of $y$ when $x = 25$. [3]



| | | | | |
|---|---|---|---|---|
| **8** | **(i)** | Sensibly separate variables and attempt integration of at least one side | M1 | |
| | | Obtain $2y^{\frac{1}{2}} = \ldots$ or equivalent | A1 | |
| | | Correct integration by parts of $x\sin\frac{1}{3}x$ as far as $ax\cos\frac{1}{3}x \pm \int b\cos\frac{1}{3}x\,dx$ | M1 | |
| | | Obtain $-3x\cos\frac{1}{3}x + \int 3\cos\frac{1}{3}x\,dx$ or equivalent | A1 | |
| | | Obtain $-3x\cos\frac{1}{3}x + 9\sin\frac{1}{3}x$ or equivalent | A1 | |
| | | Obtain $y = \left(-\frac{3}{10}x\cos\frac{1}{3}x + \frac{9}{10}\sin\frac{1}{3}x + c\right)^2$ or equivalent | A1 | [6] |
| | **(ii)** | Use $x = 0$ and $y = 100$ to find constant | M*1 | |
| | | Substitute 25 and calculate value of $y$ | DM*1 | |
| | | Obtain 203 | A1 | [3] |

10




The diagram shows part of the curve with parametric equations

$$x = 2\ln(t+2), \qquad y = t^3 + 2t + 3.$$

**(i)** Find the gradient of the curve at the origin. [5]

**(ii)** At the point $P$ on the curve, the value of the parameter is $p$. It is given that the gradient of the curve at $P$ is $\frac{1}{2}$.

**(a)** Show that $p = \dfrac{1}{3p^2+2} - 2$. [1]

**(b)** By first using an iterative formula based on the equation in part **(a)**, determine the coordinates of the point $P$. Give the result of each iteration to 5 decimal places and each coordinate of $P$ correct to 2 decimal places. [4]

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**10 (i)** Obtain $\dfrac{dx}{dt} = \dfrac{2}{t+2}$ and $\dfrac{dy}{dt} = 3t^2 + 2$

Use $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

Obtain $\dfrac{dy}{dx} = \dfrac{1}{2}(3t^2+2)(t+2)$

Identify value of $t$ at the origin as $-1$

Substitute to obtain $\dfrac{5}{2}$ as gradient at the origin

**(ii) (a)** Equate derivative to $\dfrac{1}{2}$ and confirm $p = \dfrac{1}{3p^2+2} - 2$

**(b)** Use the iterative formula correctly at least once

Obtain value $p = -1.924$ or better $(-1.92367\ldots)$

Show sufficient iterations to justify accuracy or show a sign change in appropriate interval

Obtain coordinates $(-5.15, -7.97)$

**8** The complex number $w$ is defined by $w = \dfrac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$.

**(i)** Without using a calculator, show that $w = 2 + 4\mathrm{i}$. [3]

**(ii)** It is given that $p$ is a real number such that $\frac{1}{4}\pi \leqslant \arg(w + p) \leqslant \frac{3}{4}\pi$. Find the set of possible values of $p$. [3]

**(iii)** The complex conjugate of $w$ is denoted by $w^*$. The complex numbers $w$ and $w^*$ are represented in an Argand diagram by the points $S$ and $T$ respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through $S$, $T$ and the origin. [3]



| | | | | | |
|---|---|---|---|---|---|
| **8** | **(i)** | Either | Expand $(2-\mathrm{i})^2$ to obtain $3-4\mathrm{i}$ or unsimplified equivalent | B1 | |
| | | | Multiply by $\dfrac{3+4\mathrm{i}}{3+4\mathrm{i}}$ and simplify to $x+\mathrm{i}y$ form or equivalent | M1 | |
| | | | Confirm given answer $2+4\mathrm{i}$ | A1 | |
| | | Or | Expand $(2-\mathrm{i})^2$ to obtain $3-4\mathrm{i}$ or unsimplified equivalent | B1 | |
| | | | Obtain two equations in $x$ and $y$ and solve for $x$ or $y$ | M1 | |
| | | | Confirm given answer $2+4\mathrm{i}$ | A1 | [3] |
| | **(ii)** | Identify $4+4\mathrm{i}$ or $-4+4\mathrm{i}$ as point at either end or state $p=2$ or state $p=-6$ | | B1 | |
| | | Use appropriate method to find both critical values of $p$ | | M1 | |
| | | State $-6 \leqslant p \leqslant 2$ | | A1 | [3] |
| | **(iii)** | Identify equation as of form $\lvert z-a\rvert = a$ or equivalent | | M1 | |
| | | Form correct equation for $a$ not involving modulus, e.g. $(a-2)^2+4^2=a^2$ | | A1 | |
| | | State $\lvert z-5\rvert = 5$ | | A1 | [3] |

5 **(a)** Find $\int (4 + \tan^2 2x)\,dx$. [3]

**(b)** Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x + \frac{1}{6}\pi)}{\sin x}\,dx$. [5]

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| | | | | |
|---|---|---|---|---|
| 5 | (a) | Use identity $\tan^2 2x = \sec^2 2x - 1$ | B1 | |
| | | Obtain integral of form $ax + b\tan 2x$ | M1 | |
| | | Obtain correct $3x + \frac{1}{2}\tan 2x$, condoning absence of $+ c$ | A1 | [3] |
| | (b) | State $\sin x \cos \frac{1}{2}\pi + \cos x \sin \frac{1}{6}\pi$ | B1 | |
| | | Simplify integrand to $\cos \frac{1}{6}\pi + \frac{\cos x \sin \frac{1}{6}\pi}{\sin x}$ or equivalent | B1 | |
| | | Integrate to obtain at least term of form $a\,\text{In}(\sin x)$ | *M1 | |
| | | Apply limits and simplify to obtain two terms | M1 dep *M | |
| | | Obtain $\frac{1}{8}\pi\sqrt{3} - \frac{1}{2}\ln(\frac{1}{\sqrt{2}})$ or equivalent | A1 | [5] |

**9** The complex number $3 - \mathrm{i}$ is denoted by $u$. Its complex conjugate is denoted by $u^*$.

**(i)** On an Argand diagram with origin $O$, show the points $A$, $B$ and $C$ representing the complex numbers $u$, $u^*$ and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

**(ii)** Showing your working and without using a calculator, express $\dfrac{u^*}{u}$ in the form $x + \mathrm{i}y$, where $x$ and $y$ are real. [3]

**(iii)** By considering the argument of $\dfrac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]



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| | | | | |
|---|---|---|---|---|
| **9** | **(i)** | Show $u$ in a relatively correct position | **B1** | |
| | | Show $u^*$ in a relatively correct position | **B1** | |
| | | Show $u^* - u$ in a relatively correct position | **B1** | |
| | | State or imply that $OABC$ is a parallelogram | **B1** | [4] |
| | **(ii)** | *EITHER*: Substitute for $u$ and multiply numerator and denominator by $3 + \mathrm{i}$, or equivalent | **M1** | |
| | | Simplify the numerator to $8 + 6\mathrm{i}$ or the denominator to 10 | **A1** | |
| | | Obtain final answer $\frac{4}{5} + \frac{3}{5}\mathrm{i}$, or equivalent | **A1** | |
| | | *OR*: Substitute for $u$, obtain two equations in $x$ and $y$ and solve for $x$ or for $y$ | **M1** | |
| | | Obtain $x = \frac{4}{5}$ or $y = \frac{3}{5}$, or equivalent | **A1** | |
| | | Obtain final answer $\frac{4}{5} + \frac{3}{5}\mathrm{i}$, or equivalent | **A1** | [3] |

5 The equation of a curve is $y = e^{-2x} \tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

**(i)** Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where $a$ and $b$ are constants. [5]

**(ii)** Explain why the gradient of the curve is never negative. [1]

**(iii)** Find the value of $x$ for which the gradient is least. [1]

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| | | | | |
|---|---|---|---|---|
| 5 | (i) | State or imply that the derivative of $e^{-2x}$ is $-2e^{-2x}$ | **B1** | |
| | | Use product or quotient rule | **M1** | |
| | | Obtain correct derivative in any form | **A1** | |
| | | Use Pythagoras | **M1** | |
| | | Justify the given form | **A1** | [5] |
| | (ii) | Fully justify the given statement | **B1** | [1] |
| | (iii) | State answer $x = \frac{1}{4}\pi$ | **B1** | [1] |

**3** The angles $\theta$ and $\phi$ lie between 0° and 180°, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of $\theta$ and $\phi$. [6]

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| | | | |
|---|---|---|---|
| **3** | Use $\tan(A \pm B)$ and obtain an equation in $\tan\theta$ and $\tan\phi$ | **M1*** | |
| | Substitute throughout for $\tan\theta$ or for $\tan\phi$ | **dep M1*** | |
| | Obtain $3\tan^2\theta - \tan\theta - 4 = 0$ or $3\tan^2\phi - 5\tan\phi - 2 = 0$, or 3-term equivalent | **A1** | |
| | Solve a 3-term quadratic and find an angle | **M1** | |
| | Obtain answer $\theta = 135°$, $\phi = 63.4°$ | **A1** | |
| | Obtain answer $\theta = 53.1°$, $\phi = 161.6°$ | **A1** | [6] |
| | [Treat answers in radians as a misread. Ignore answers outside the given interval.] | | |
| | [SR: Two correct values of $\theta$ (or $\phi$) score A1; then A1 for both correct $\theta$, $\phi$ pairs.] | | |

**9** **(a)** It is given that $(1 + 3\mathrm{i})w = 2 + 4\mathrm{i}$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

**(b)** On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $re^{\mathrm{i}\theta}$. [4]



| | | | | |
|---|---|---|---|---|
| **9** | **(a)** | Either | Find $w$ using conjugate of $1+3\mathrm{i}$ | **M1** |
| | | | Obtain $\frac{7-\mathrm{i}}{5}$ or equivalent | **A1** |
| | | | Square $x+\mathrm{i}y$ form to find $w^2$ | **M1** |
| | | | Obtain $w^2 = \frac{48-14\mathrm{i}}{25}$ and confirm modulus is 2 | **A1** |
| | | | Use correct process for finding argument of $w^2$ | **M1** |
| | | | Obtain –0.284 radians or $-16.3°$ | **A1** |
| | | Or 1 | Find $w$ using conjugate of $1+3\mathrm{i}$ | **M1** |
| | | | Obtain $\frac{7-\mathrm{i}}{5}$ or equivalent | **A1** |
| | | | Find modulus of $w$ and hence of $w^2$ | **M1** |
| | | | Confirm modulus is 2 | **A1** |
| | | | Find argument of $w$ and hence of $w^2$ | **M1** |
| | | | Obtain –0.284 radians or $-16.3°$ | **A1** |
| | | Or 2 | Square both sides to obtain $(-8+6\mathrm{i})w^2 = -12+16\mathrm{i}$ | **B1** |
| | | | Find $w^2$ using relevant conjugate | **M1** |
| | | | Use correct process for finding modulus of $w^2$ | **M1** |
| | | | Confirm modulus is 2 | **A1** |
| | | | Use correct process for finding argument of $w^2$ | **M1** |
| | | | Obtain –0.284 radians or $-16.3°$ | **A1** |

7 (i) Show that $(x + 1)$ is a factor of $4x^3 - x^2 - 11x - 6$. [2]

(ii) Find $\int \frac{4x^2 + 9x - 1}{4x^3 - x^2 - 11x - 6} \, dx$. [8]

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| | | | | | |
|---|---|---|---|---|---|
| 7 | (i) | Either | Substitute $x = -1$ and evaluate | M1 | |
| | | | Obtain 0 and conclude $x + 1$ is a factor | A1 | |
| | | Or | Divide by $x + 1$ and obtain a constant remainder | M1 | |
| | | | Obtain remainder = 0 and conclude $x + 1$ is a factor | A1 | [2] |
| | (ii) | | Attempt division, or equivalent, at least as far as quotient $4x^2 + kx$ | M1 | |
| | | | Obtain complete quotient $4x^2 - 5x - 6$ | A1 | |
| | | | State form $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{4x+3}$ | A1 | |
| | | | Use relevant method for finding at least one constant | M1 | |
| | | | Obtain one of $A = -2, B = 1, C = 8$ | A1 | |
| | | | Obtain all three values | A1 | |
| | | | Integrate to obtain three terms each involving natural logarithm of linear form | M1 | |
| | | | Obtain $-2\ln(x+1) + \ln(x-2) + 2\ln(4x+3)$, condoning no use of modulus signs and absence of $\ldots + c$ | A1 | [8] |

**6** The angles $A$ and $B$ are such that

$$\sin(A + 45°) = (2\sqrt{2})\cos A \quad \text{and} \quad 4\sec^2 B + 5 = 12\tan B.$$

Without using a calculator, find the exact value of $\tan(A - B)$. [8]

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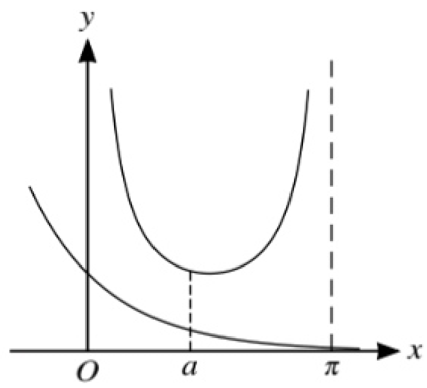
| | | | |
|---|---|---|---|
| **6** | State or imply $\sin A \times \cos 45 + \cos A \times \sin 45 = 2\sqrt{2}\cos A$ | **B1** | |
| | Divide by $\cos A$ to find value of $\tan A$ | **M1** | |
| | Obtain $\tan A = 3$ | **A1** | |
| | Use identity $\sec^2 B = 1 + \tan^2 B$ | **B1** | |
| | Solve three-term quadratic equation and find $\tan B$ | **M1** | |
| | Obtain $\tan B = \frac{3}{2}$ only | **A1** | |
| | Substitute **numerical values** in $\frac{\tan A - \tan B}{1 + \tan A \tan B}$ | **M1** | |
| | Obtain $\frac{3}{11}$ | **A1** | [8] |

**8**




The diagram shows the curve $y = \operatorname{cosec} x$ for $0 < x < \pi$ and part of the curve $y = e^{-x}$. When $x = a$, the tangents to the curves are parallel.

**(i)** By differentiating $\dfrac{1}{\sin x}$, show that if $y = \operatorname{cosec} x$ then $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$. [3]

**(ii)** By equating the gradients of the curves at $x = a$, show that

$$a = \tan^{-1}\left(\frac{e^{a}}{\sin a}\right).$$ [2]

**(iii)** Verify by calculation that $a$ lies between 1 and 1.5. [2]

**(iv)** Use an iterative formula based on the equation in part **(ii)** to determine $a$ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]



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| | | | | |
|---|---|---|---|---|
| **8** | **(i)** | Use correct quotient or chain rule | **M1** | |
| | | Obtain correct derivative in any form | **A1** | |
| | | Obtain the given answer correctly | **A1** | [3] |
| | **(ii)** | State a correct equation, e.g. $-e^{-a} = -\operatorname{cosec} a \cot a$ | **B1** | |
| | | Rearrange it correctly in the given form | **B1** | [2] |
| | **(iii)** | Calculate values of a relevant expression or pair of expressions at $x = 1$ and $x = 1.5$ | **M1** | |
| | | Complete the argument correctly with correct calculated values | **A1** | [2] |
| | **(iv)** | Use the iterative formula correctly at least once | **M1** | |
| | | Obtain final answer 1.317 | **A1** | |
| | | Show sufficient iterations to 5 d.p. to justify 1.317 to 3 d.p., or show there is a sign change in the interval (1.3165, 1,3175) | **A1** | [3] |

5 **(i)** Prove the identity $\cos 4\theta - 4\cos 2\theta \equiv 8\sin^4\theta - 3$. [4]

**(ii)** Hence solve the equation

$$\cos 4\theta = 4\cos 2\theta + 3,$$

for $0° \leqslant \theta \leqslant 360°$. [4]

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| | | | | |
|---|---|---|---|---|
| 5 | (i) | *EITHER*: Express $\cos 4\theta$ in terms of $\cos 2\theta$ and/or $\sin 2\theta$ | **B1** | |
| | | Use correct double angle formulae to express LHS in terms of $\sin\theta$ and/or $\cos\theta$ | **M1** | |
| | | Obtain a correct expression in terms of $\sin\theta$ alone | **A1** | |
| | | Reduce correctly to the given form | **A1** | |
| | | *OR*: Use correct double angle formula to express RHS in terms of $\cos 2\theta$ | **M1** | |
| | | Express $\cos^2 2\theta$ in terms of $\cos 4\theta$ | **B1** | |
| | | Obtain a correct expression in terms of $\cos 4\theta$ and $\cos 2\theta$ | **A1** | |
| | | Reduce correctly to the given form | **A1** | [4] |
| | (ii) | Use the identity and carry out a method for finding a root | **M1** | |
| | | Obtain answer 68.5° | **A1** | |
| | | Obtain a second answer, e.g. 291.5° | **A1**√ | |
| | | Obtain the remaining answers, e.g. 111.5° and 248.5°, and no others in the given interval | **A1**√ | [4] |
| | | [Ignore answers outside the given interval. Treat answers in radians as a misread.] | | |

**10** The polynomial $\mathrm{p}(z)$ is defined by

$$\mathrm{p}(z) = z^3 + mz^2 + 24z + 32,$$

where $m$ is a constant. It is given that $(z + 2)$ is a factor of $\mathrm{p}(z)$.

**(i)** Find the value of $m$. [2]

**(ii)** Hence, showing all your working, find

**(a)** the three roots of the equation $\mathrm{p}(z) = 0$, [5]

**(b)** the six roots of the equation $\mathrm{p}(z^2) = 0$. [6]

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| | | | |
|---|---|---|---|
| **10 (i)** | Attempt to solve for $m$ the equation $\mathrm{p}(-2) = 0$ or equivalent | M1 | |
| | Obtain $m = 6$ | A1 | [2] |
| | *Alternative*: | | |
| | Attempt $\mathrm{p}(z) \div (z + 2)$, equate a constant remainder to zero and solve for $m$. | M1 | |
| | Obtain $m = 6$ | A1 | |
| **(ii) (a)** | State $z = -2$ | B1 | |
| | Attempt to find quadratic factor by inspection, division, identity, … | M1 | |
| | Obtain $z^2 + 4z + 16$ | A1 | |
| | Use correct method to solve a 3-term quadratic equation | M1 | |
| | Obtain $-2 \pm 2\sqrt{3}\mathrm{i}$ or equivalent | A1 | [5] |
| **(b)** | State or imply that square roots of answers from part **(ii)(a)** needed | M1 | |
| | Obtain $\pm \mathrm{i}\sqrt{2}$ | A1 | |
| | Attempt to find square root of a further root in the form $x + \mathrm{i}y$ or in polar form | M1 | |
| | Obtain $a^2 - b^2 = -2$ and $ab = (\pm)\sqrt{3}$ following their answer to part **(ii)(a)** | A1√ | |
| | Solve for $a$ and $b$ | M1 | |
| | Obtain $\pm\left(1 + \mathrm{i}\sqrt{3}\right)$ and $\pm\left(1 - \mathrm{i}\sqrt{3}\right)$ | A1 | [6] |

8 (i) Express $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of $\alpha$ correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle $\theta$ which satisfies the equation

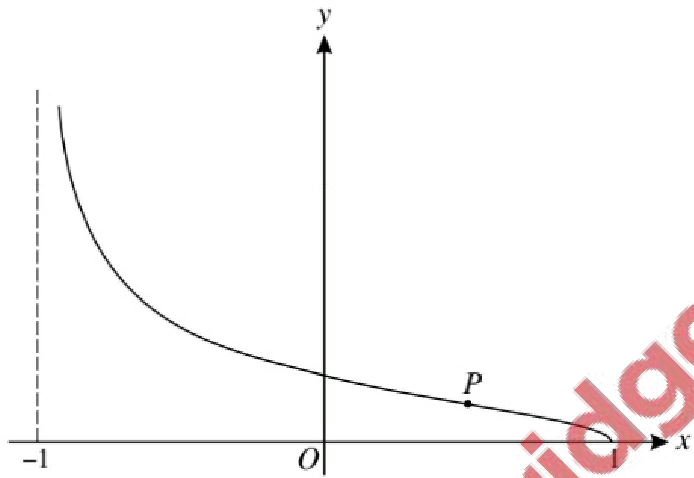
(a) $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta = -4$, [2]

(b) $(\sqrt{6})\cos\frac{1}{2}\theta + (\sqrt{10})\sin\frac{1}{2}\theta = 3$. [4]

9709_w10_ms_33.pdf

| | | | | |
|---|---|---|---|---|
| 8 | (i) | Obtain or imply $R = 4$ | B1 | |
| | | Use appropriate trigonometry to find $\alpha$ | M1 | |
| | | Obtain $\alpha = 52.24$ or better from correct work | A1 | [3] |
| | (ii) | (a) State or imply $\theta - \alpha = \cos^{-1}(-4 \div R)$ | M1 | |
| | | Obtain 232.2 or better | A1 | [2] |
| | | (b) Attempt at least one value using $\cos^{-1}(3 \div R)$ | M1 | |
| | | Obtain one correct value e.g. $\pm 41.41^\circ$ | A1 | |
| | | Use $\frac{1}{2}\theta - \alpha = \cos^{-1}\left(\frac{3}{R}\right)$ to find $\theta$ | M1 | |
| | | Obtain 21.7 | A1 | [4] |

**9**




The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

**(i)** By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of $x$. Hence show that the gradient of the normal to the curve at the point $(x, y)$ is $(1 + x)\sqrt{(1 - x^2)}$. [5]

**(ii)** The gradient of the normal to the curve has its maximum value at the point $P$ shown in the diagram. Find, by differentiation, the $x$-coordinate of $P$. [4]



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| | | | | |
|---|---|---|---|---|
| **9** | **(i)** | Use quotient or product rule to differentiate $(1 - x)/(1 + x)$ | M1 | |
| | | Obtain correct derivative in any form | A1 | |
| | | Use chain rule to find $\frac{dy}{dx}$ | M1 | |
| | | Obtain a correct expression in any form | A1 | |
| | | Obtain the gradient of the normal in the given form correctly | A1 | [5] |
| | **(ii)** | Use product rule | M1 | |
| | | Obtain correct derivative in any form | A1 | |
| | | Equate derivative to zero and solve for $x$ | M1 | |
| | | Obtain $x = \frac{1}{2}$ | A1 | [4] |

5 The curve with equation

$$6e^{2x} + ke^{y} + e^{2y} = c,$$

where $k$ and $c$ are constants, passes through the point $P$ with coordinates $(\ln 3, \ln 2)$.

**(i)** Show that $58 + 2k = c$. [2]

**(ii)** Given also that the gradient of the curve at $P$ is $-6$, find the values of $k$ and $c$. [5]



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| | | | | |
|---|---|---|---|---|
| 5 | (i) | Use at least one of $e^{2x} = 9$, $e^{y} = 2$ and $e^{2y} = 4$ | B1 | |
| | | Obtain given result $58 + 2k = c$ AG | B1 | [2] |
| | (ii) | Differentiate left-hand side term by term, reaching $ae^{2x} + be^{y}\frac{dy}{dx} + ce^{2y}\frac{dy}{dx}$ | M1 | |
| | | Obtain $12e^{2x} + ke^{y}\frac{dy}{dx} + 2e^{2y}\frac{dy}{dx}$ | A1 | |
| | | Substitute $(\ln 3, \ln 2)$ in an attempt involving implicit differentiation at least once, where RHS = 0 | M1 | |
| | | Obtain $108 - 12k - 48 = 0$ or equivalent | A1 | |
| | | Obtain $k = 5$ and $c = 68$ | A1 | [5] |

4 The polynomial $f(x)$ is defined by

$$f(x) = 12x^3 + 25x^2 - 4x - 12.$$

**(i)** Show that $f(-2) = 0$ and factorise $f(x)$ completely. [4]

**(ii)** Given that

$$12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 = 0,$$

state the value of $3^y$ and hence find $y$ correct to 3 significant figures. [3]

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| | | | | |
|---|---|---|---|---|
| 4 | (i) | Verify that $-96 + 100 + 8 - 12 = 0$ | B1 | |
| | | Attempt to find quadratic factor by division by $(x + 2)$, reaching a partial quotient $12x^2 + kx$, inspection or use of an identity | M1 | |
| | | Obtain $12x^2 + x - 6$ | A1 | |
| | | State $(x + 2)(4x + 3)(3x - 2)$ | A1 | [4] |
| | | [The M1 can be earned if inspection has unknown factor $Ax^2 + Bx - 6$ and an equation in $A$ and/or $B$ or equation $12x^2 + Bx + C$ and an equation in $B$ and/or $C$.] | | |
| | (ii) | State $3^y = \frac{2}{3}$ and no other value | B1 | |
| | | Use correct method for finding $y$ from equation of form $3^y = k$, where $k > 0$ | M1 | |
| | | Obtain $-0.369$ and no other value | A1 | [3] |

9 In a chemical reaction, a compound $X$ is formed from two compounds $Y$ and $Z$. The masses in grams of $X$, $Y$ and $Z$ present at time $t$ seconds after the start of the reaction are $x$, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of $X$ is proportional to the product of the masses of $Y$ and $Z$ present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

**(i)** Show that $x$ and $t$ satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

**(ii)** Solve this differential equation and obtain an expression for $x$ in terms of $t$. [9]

**(iii)** State what happens to the value of $x$ when $t$ becomes large. [1]

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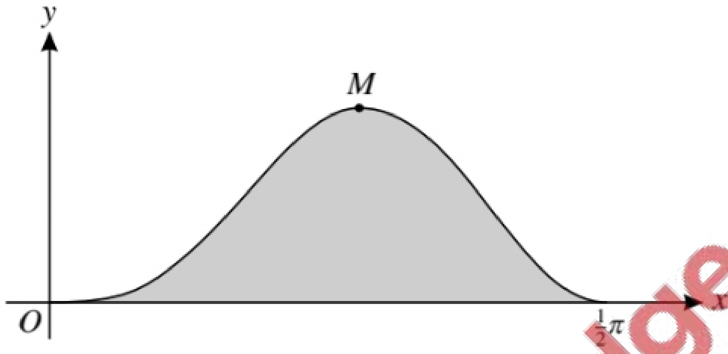
| | | | |
|---|---|---|---|
| 9 (i) | State or imply $\frac{dx}{dt} = k(10 - x)(20 - x)$ and show $k = 0.01$ | B1 | [1] |
| (ii) | Separate variables correctly and attempt integration of at least one side | M1 | |
| | Carry out an attempt to find $A$ and $B$ such that $\frac{1}{(10-x)(20-x)} \equiv \frac{A}{10-x} + \frac{B}{20-x}$, or equivalent | M1 | |
| | Obtain $A = \frac{1}{10}$ and $B = -\frac{1}{10}$, or equivalent | A1 | |
| | Integrate and obtain $-\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x)$, or equivalent | A1√ | |
| | Integrate and obtain term $0.01t$, or equivalent | A1 | |
| | Evaluate a constant, or use limits $t = 0$, $x = 0$, in a solution containing terms of the form $a\ln(10-x)$, $b\ln(20-x)$ and $ct$ | M1 | |
| | Obtain answer in any form, e.g. $-\frac{1}{10}\ln(10-x) + \frac{1}{10}\ln(20-x) = 0.01t + \frac{1}{10}\ln 2$ | A1√ | |
| | Use laws of logarithms to correctly remove logarithms | M1 | |
| | Rearrange and obtain $x = 20(\exp(0.1t) - 1)/(2\exp(0.1t) - 1)$, or equivalent | A1 | [9] |
| (iii) | State that $x$ approaches 10 | B1 | [1] |

**8**




The diagram shows the curve $y = 5\sin^3 x\cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point $M$.

**(i)** Find the $x$-coordinate of $M$. [5]

**(ii)** Using the substitution $u = \cos x$, find by integration the area of the shaded region bounded by the curve and the $x$-axis. [5]



| | | | | |
|---|---|---|---|---|
| 8 | (i) | Use product and chain rule | M1 | |
| | | Obtain correct derivative in any form, e.g. $15\sin^2 x\cos^3 x - 10\sin^4 x\cos x$ | A1 | |
| | | Equate derivative to zero and obtain a relevant equation in one trigonometric function | M1 | |
| | | Obtain $2\tan^2 x = 3$, $5\cos^2 x = 2$, or $5\sin^2 x = 3$ | A1 | |
| | | Obtain answer $x = 0.886$ radians | A1 | [5] |
| | (ii) | State or imply $\mathrm{d}u = -\sin x\,\mathrm{d}x$, or $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$, or equivalent | B1 | |
| | | Express integral in terms of $u$ and $\mathrm{d}u$ | M1 | |
| | | Obtain $\pm\int 5(u^2 - u^4)\,\mathrm{d}u$, or equivalent | A1 | |
| | | Integrate and use limits $u = 1$ and $u = 0$ (or $x = 0$ and $x = \frac{1}{2}\pi$) | M1 | |
| | | Obtain answer $\frac{2}{3}$, or equivalent, with no errors seen | A1 | [5] |

7 **(i)** Find the roots of the equation

$$z^2 + (2\sqrt{3})z + 4 = 0,$$

giving your answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real. [2]

**(ii)** State the modulus and argument of each root. [3]

**(iii)** Showing all your working, verify that each root also satisfies the equation

$$z^6 = -64.$$ [3]



7 **(i)** Use the quadratic formula, completing the square, or the substitution $z = x + \mathrm{i}y$ to find a root and use $\mathrm{i}^2 = -1$ — M1

Obtain final answers $-\sqrt{3} \pm \mathrm{i}$, or equivalent — A1 [2]

**(ii)** State that the modulus of both roots is 2 — B1√

State that the argument of $-\sqrt{3} + \mathrm{i}$ is 150° or $\frac{5}{6}\pi$ (2.62) radians — B1√

State that the argument of $-\sqrt{3} - \mathrm{i}$ is –150° (or 210°) or $-\frac{5}{6}\pi$ (–2.62) radians or $\frac{7}{6}\pi$ (3.67) radians — B1√ [3]

**(iii)** Carry out an attempt to find the sixth power of a root — M1

Verify that one of the roots satisfies $z^6 = -64$ — A1

Verify that the other root satisfies the equation — A1 [3]

**6** Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

**(i)** Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1}\,du$. [3]

**(ii)** Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

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| | | | | |
|---|---|---|---|---|
| **6** | **(i)** | State or imply $du = \frac{1}{2\sqrt{x}}dx$ | **B1** | |
| | | Substitute for $x$ and $dx$ throughout | **M1** | |
| | | Justify the change in limits and obtain the given answer | **A1** | [3] |
| | **(ii)** | Convert integrand into the form $A + \frac{B}{u+1}$ | **M1*** | |
| | | Obtain integrand $A = 1$, $B = -2$ | **A1** | |
| | | Integrate and obtain $u - 2\ln(u+1)$ | **A1√ + A1√** | |
| | | Substitute limits correctly in an integral containing terms $au$ and $b\ln(u+1)$, where $ab \neq 0$ | **DM1** | |
| | | Obtain the given answer following full and correct working | **A1** | [6] |
| | | [The f.t. is on $A$ and $B$.] | | |

**10** **(i)** It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting $\tan x$ by $t$, form an equation in $t$ and hence show that either $t = 0$ or $t = \sqrt[3]{(t + 0.8)}$. [4]

**(ii)** It is given that there is exactly one real value of $t$ satisfying the equation $t = \sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

**(iii)** Use the iterative formula $t_{n+1} = \sqrt[3]{(t_n + 0.8)}$ to find the value of $t$ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

**(iv)** Using the values of $t$ found in previous parts of the question, solve the equation

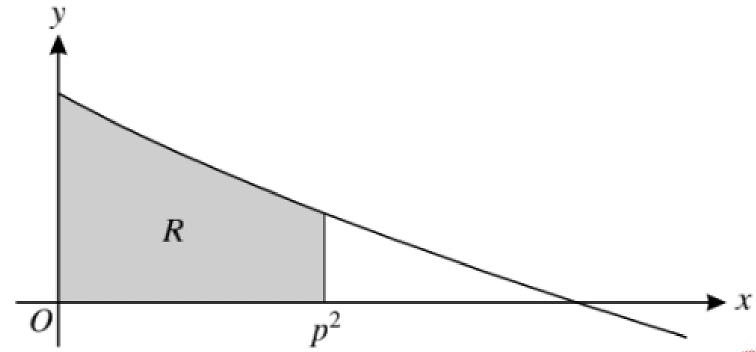
$$2\tan 2x + 5\tan^2 x = 0$$

for $-\pi \leqslant x \leqslant \pi$. [3]



| | | | | |
|---|---|---|---|---|
| **10** | **(i)** | Use correct identity for $\tan 2x$ and obtains $at^4 + bt^3 + ct^2 + dt = 0$, where $b$ may be zero | M1 | |
| | | Obtain correct horizontal equation, e.g. $4t + 5t^2 - 5t^4 = 0$ | A1 | |
| | | Obtain $kt(t^3 + et + f) = 0$ or equivalent | M1 | |
| | | Confirm given results $t = 0$ and $t = \sqrt[3]{t + 0.8}$ | A1 | [4] |
| | **(ii)** | Consider sign of $t - \sqrt[3]{t + 0.8}$ at 1.2 and 1.3 or equivalent | M1 | |
| | | Justify the given statement with correct calculations (–0.06 and 0.02) | A1 | [2] |
| | **(iii)** | Use the iterative formula correctly at least once with $1.2 < t_n < 1.3$ | M1 | |
| | | Obtain final answer 1.276 | A1 | |
| | | Show sufficient iterations to justify answer or show there is a change of sign in interval (1.2755, 1.2765) | A1 | [3] |
| | **(iv)** | Evaluate $\tan^{-1}$ (answer from part **(iii)**) to obtain at least one value | M1 | |
| | | Obtain –2.24 and 0.906 | A1 | |
| | | State $-\pi$, 0 and $\pi$ | B1 | [3] |
| | | [SR If A0, B0, allow B1 for any 3 roots] | | |

**7**




The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where $x$ is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by $R$. The area of $R$ is equal to 1.

**(i)** Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \dfrac{3 - 2\cos p}{2p}$. [6]

**(ii)** Use the iterative formula $p_{n+1} = \sin^{-1}\left(\dfrac{3 - 2\cos p_n}{2p_n}\right)$, with initial value $p_1 = 1$, to find the value of $p$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]



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| | | | | |
|---|---|---|---|---|
| 7 | (i) | Substitute for $x$ and $dx$ throughout the integral | M1 | |
| | | Obtain $\int 2u\cos u\,du$ | A1 | |
| | | Integrate by parts and obtain answer of the form $au\sin u + b\cos u$, where $ab \neq 0$ | M1 | |
| | | Obtain $2u\sin u + 2\cos u$ | A1 | |
| | | Use limits $u = 0$, $u = p$ correctly and equate result to 1 | M1 | |
| | | Obtain the given answer | A1 | [6] |
| | (ii) | Use the iterative formula correctly at least once | M1 | |
| | | Obtain final answer $p = 1.25$ | A1 | |
| | | Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.245, 1.255) | A1 | [3] |

**6** It is given that $\tan 3x = k \tan x$, where $k$ is a constant and $\tan x \neq 0$.

**(i)** By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

**(ii)** Hence solve the equation $\tan 3x = k \tan x$ when $k = 4$, giving all solutions in the interval $0^\circ < x < 180^\circ$. [3]

**(iii)** Show that the equation $\tan 3x = k \tan x$ has no root in the interval $0^\circ < x < 180^\circ$ when $k = 2$. [1]

## 9709_s12_ms_33.pdf



| | | | | |
|---|---|---|---|---|
| **6** | **(i)** | Use $\tan(A + B)$ and $\tan 2A$ formulae to obtain an equation in $\tan x$ | M1 | |
| | | Obtain a correct equation in $\tan x$ in any form | A1 | |
| | | Obtain an expression of the form $a \tan^2 x = b$ | M1 | |
| | | Obtain the given answer | A1 | [4] |
| | **(ii)** | Substitute $k = 4$ in the given expression and solve for $x$ | M1 | |
| | | Obtain answer, e.g. $x = 16.8°$ | A1 | |
| | | Obtain second answer, e.g. $x = 163.2°$, and no others in the given interval | A1 | [3] |
| | | [Ignore answers outside the given interval. Treat answers in radians as a misread and deduct A1 from the marks for the angles.] | | |
| | **(iii)** | Substitute $k = 2$, show $\tan^2 x < 0$ and justify given statement correctly | B1 | [1] |