FURTHER MATHEMATICS

Paper 1348/01

Further Pure Mathematics

Key messages

Candidates need to be more willing to give full explanations and justifications when these are required. They also need to consider the conciseness of their solutions to ensure that they do not run out of time to complete the paper.

General Comments

The standard of the work presented by the candidates was very good, with many achieving a three-figure total. In this paper, candidates were often being required to grasp the underlying ideas throughout a question, rather than being led through towards the solution. This meant that some of the shorter questions, in particular, proved to be quite tricky. **Questions 3, 7** and **10** were especially successful questions, while **Questions 11** and **12** were also popular sources of marks. **Question 13**, being the final question on the paper was often left incomplete, while **Question 6** was found difficult due to the fact that it required a firm grasp of the "big picture" in order for successful progress to be made. The most obvious general weakness amongst the candidature was the lack of enthusiasm for providing explanations and justifications; thus **Question 8**, a "theoretical" question on the group theory work, proved to be difficult for candidates. The other major hurdle to a complete attempt at all questions was the frequent deployment of very lengthy working in order to establish some very straightforward, routine tasks. For example, the bookwork at the start of **Question 13**, which often elicited up to two pages of working when a few lines should have been sufficient. The need for conciseness is paramount in these papers in order to allow candidates sufficient time to check their work.

Comments on specific questions

Question 1

This was a very gentle starter, and it was successfully attempted by almost all candidates. A lack of care when dealing with the details, such as in the handling of the constants involved in the "translated" standard integral or in dealing with the subtracted negative angle, resulted in some candidates not scoring full marks. A very small number of candidates lost the final mark by working in degrees.

Answer: $\frac{\sqrt{3}}{6}\pi$

Question 2

This was another straightforward question. However, none of the candidates gained all 4 marks on this question, as the majority were unaware of the requirement to justify the stated domain. A small number of candidates *did* realise that they should be addressing this issue, but they then did not do so satisfactorily. The series for $\ln(1 + x)$ is given in the List of Formulae, although some candidates ignored the prompt given in the question. They then had to amend this series for $\ln(1 – x)$, tidy up the difference of the two series with a bit of logarithmic work and point out that the answer was equal to the given series for $\text{arctanh}(x)$. However, there was also the one mark for noting, *explicitly*, that the domains for $\ln(1 + x)$ and $\ln(1 – x)$ needed to be "trimmed" slightly in order to make the stated result valid.

Question 3

Candidates clearly feel quite comfortable with the demands of the topic of rational functions, and marks scored on this question were high. However, very few scored all of the 9 marks available. As mentioned in the general comments, explanations and justifications were generally found insufficient, and the modest bit of work to show *convincingly* that the gradient of the given curve was indeed "always negative" was widely flawed, usually consisting of a comment to the effect that the numerator of the derivative consisted exclusively of "minus" terms.

Another small detail that might help future candidates is that it is expected that they will **note** "significant features" such as the coordinates of key points and the equations of asymptotes. On this question, where these were requested only to be shown, this was not so important and candidates were rewarded fully irrespective of whether they had shown the features on their sketch or noted them in writing alongside. However, it would be best if they were to be clear in what they wrote; for instance, rather than noting the asymptote "$y = 0$", many just drew it, while others stated it in "behavioural" forms such as "$y \to 0$" or "$y \to \frac{1}{x}$".

Question 4

This was a relatively routine vectors question, with part **(ii)** requiring little more than the use of a standard result. Those candidates who felt obliged to work out the shortest distance, by first determining the endpoints of the mutual perpendicular, made this question tougher than was intended.

Answers: **(i)** any (non-zero) multiple of $2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ **(ii)** $\sqrt{38}$

Question 5

Part **(i)** of this question was found to be very routine and, although part **(ii)** was also quite routine in principle, quite a few candidates seemed unable to approach the relevant use of *de Moivre's theorem* in this reversed way; that is, they seemed to know how to employ the work from part **(i)** to get, for example, $\sin 5\theta$ in terms of powers of $\sin\theta$, but were uncertain as to what was required in order to get a power of $\sin\theta$ in terms of the sines of multiple angles. These candidates apart, the biggest obstacle to a completely successful solution turned out to be the inability to keep the i's when using the result of part **(i)**; these often just disappeared altogether.

Answers: **(ii)** $A = \frac{1}{16}$, $B = -\frac{5}{16}$, $C = \frac{5}{8}$

Question 6

As mentioned in the introduction, this was found to be one of the more difficult questions on the paper. In some ways, this was a natural consequence of its formulation: those candidates who could not complete part **(i)** correctly were unable to produce the ensuing sketch in part **(ii)**. The result of not realising what was going on in both part **(i)** and part **(ii)** necessarily meant that the integral requested in part **(iii)** was most unlikely to be referenced to a region of the cartesian plane which would then have been very easily found. Many of those candidates who did derive the correct cartesian equation in part **(i)** still did not plot the graph of a simple parabola, adopting instead the approach of plotting individual points for the curve in its polar form; such sketches almost invariably curled back on themselves or formed a closed curve.

Answers: **(i)** $y = \frac{1}{2}(x^2 - 1)$ **(iii)** $\frac{4}{3}$

Question 7

This was the most successfully answered question on the paper. The most important part of the success that followed lay in the ability to factorise the sum of two cubes in part **(i)**; fortunately, almost all candidates found this straightforward and thus progressed through the question very comfortably.

Answers: **(i)** $(x + y)^3 - 3xy(x + y)$ **(ii) (a)** 19 **(b)** $19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$

Question 8

This group theory question was the least popular question on the paper, with around one in six candidates omitting it completely and, on average, it scored the poorest of all the questions. Lots of spurious reasons were given to justify the required results, with candidates clearly not sure how to write their justifications. In part **(i)(a)**, it was clear that many candidates did not realise the significance of the double-headed implication arrow, with many giving the result in only the one direction. Since one direction of the argument is trivial, this made the mark for it almost a giveaway, yet one which many candidates did not attempt. The simplest answer to part **(b)** would be the observation that $x\{G\}$ represents a row of the group table and is thus a permutation of the elements of G (by the *Latin Square* principle). The intended answer, however, was that candidates would realise, using the result of part **(i)(a)**, that each of the n xg_is were distinct elements of G and hence the full list of them merely formed a re-ordering of G's elements.

In part **(ii)**, despite the clear hint (and the near presence of the immediately preceding result), most efforts performed only one product of all of G's elements rather than the intended two. The other crucial, usually missing, component of a successful argument here was to observe the existence of an inverse element for each member of the group. Part **(iii)(a)** only required that candidates should note, by *Lagrange's theorem*, that elements can have an order which is a factor of n; and part **(b)** only required the observation that the result of part **(ii)** had relied on the *Abelian* nature of G.

Question 9

As with **Question 8**, many candidates omitted this question. For those candidates attempting the question, success revolved around how well they could write down the four matrices involved: the key phrase there being "write down". Although a bit of thought is needed to pin down the matrices for the two shears, many candidates spent their time working out from scratch those for the reflection and rotation, rather than using the List of Formulae and putting in the relevant numbers.

Most candidates who got this far figured out the correct order in which to multiply these four matrices; but many of these attempts foundered, however, when candidates kept the factors of $\frac{1}{\sqrt{2}}$ in each entry of the reflection and rotation matrices instead of taking them out as common factors (and multiplying them together to get $\frac{1}{2}$). All these obstacles meant that fully correct final matrices, and following descriptions, were rather few.

Answers: $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection in $y = x$

Question 10

This was the first of the four longer, structured, questions on the paper, and these questions brought most candidates a substantial proportion of their marks. The routine nature of part **(a)** was attempted by all candidates, and most of them successfully reached the end correctly. A few, however, left the complementary function (CF) in complex number form, which is not the accepted way of expressing the answer. Candidates should be encouraged to quote standard results, in this case, writing down the usual form of the CF when the auxiliary equation (AE) has complex roots, rather than deriving the CF by doing the background bookwork of substituting into the given differential equation solutions of the form Ae^{kx} in order to obtain the AE. For these second-order differential equations, it is expected that the AE will be written straight down, solved, and then the CF deduced immediately without any need to resort to any background justification.

Part **(ii)** was also reasonably routine and so was similarly well-received. The only common issue arose in the differentiation of the product term $y^2 \dfrac{dy}{dx}$, where there should have been a squared $\dfrac{dy}{dx}$ term which often appeared only singly.

Answers: **(a)** $k = -2$ $y = (A - 2x)\cos x + B\sin x$ **(b) (i)** 78 **(ii)** 2.013

Question 11

Part **(i)** was found straightforward by the majority of candidates, although many forgot to find two solutions to any of the possible resulting quadratics and one or two did not note that the positive value for p went with the negative value of q (and vice versa). Part **(ii)(a)** was intended to be little more than a simple bit of work on the relationships between coefficients and roots of the cubic equation; most candidates treated this part in this very simple way while others went about things in much more extended ways.

Part **(ii)(b)** was the only part of the question that quite a lot of candidates found difficult. Most of these candidates went about things in some unnecessarily lengthy ways, principally by solving the quadratic equation by setting up more relationships between roots and coefficients and then working with these, rather than just by solving the quadratic using the formula. It was hoped that candidates would reach this stage and appreciate the unity of the whole question, realising that the solution to part **(i)** was embedded in this final part of the question. However, very few candidates realised that this was the case: those that did finished the question quickly and efficiently, the rest usually struggled lengthily and with little ultimate success.

Answers: **(i)** $p = \pm 8$, $q = \mp 1$; **(ii) (a)** $A = 4 - 4\text{i}$, $B = 21 - 16\text{i}$, $C = 84$; **(b)** $1 - 4\text{i}$, $\frac{5}{3} + \frac{4}{3}\text{i}$.

Question 12

Induction questions are generally found to be difficult, and this was no exception. Nonetheless, despite the inherent difficulties associated with this kind of proof, very few candidates were not awarded the first four marks for obtaining the results for the cases $n = 1$ to 4 and almost all then spotted that the coefficient of x in the given linear multiple of e^{2x} was simply the corresponding power of 2. The constant term in this part of the required expression proved to be the big hindrance, and many gave up at this point. A few candidates carried on without it and gained a couple of the following marks. However, the very final mark, allocated to a completely satisfactory round-up of the inductive proof, was seldom gained, as most candidates resorted to a "QED" type of approach rather than give any explanation. The problem is that the "result for $n = k + 1$" needs to be ***shown*** to be of the right form, either by working out what it should be in advance or by writing it in the right form explicitly – in this case, $\dfrac{\text{d}^{n+1}y}{\text{d}x^{n+1}} = (2^{n+1}x + (n+1).2^{(n+1)-1})\,\text{e}^{2x}$.

Answers: **(i)** $(2x + 1)\text{e}^{2x}$, $(4x + 4)\text{e}^{2x}$, $(8x + 12)\text{e}^{2x}$, $(16x + 32)\text{e}^{2x}$. **(ii)** $(2^n x + n2^{n-1})\text{e}^{2x}$.

Question 13

After a straightforward opening, this was definitely a hard question, differentiating well between the candidates. It general, candidates did an unnecessary amount of work in proving both of the results in part **(i)**, though most attempts picked up 3 or 4 of the marks available. Part **(ii)** should have been attempted more readily by more candidates, especially as the given result can easily be established in either one of two ways: firstly by taking the direct hint in the question and looking at the single integral obtained when working with the given expression, $I_{n-1} - I_n$; alternatively by splitting up the terms of the given integral I_n appropriately and integrating by parts. Once again, many candidates did not look for any connection between the different parts of the question, and this second approach was facilitated by employing the identity of part **(i)(a)**.

The allocation of only 1 mark to part **(ii)(b)** should have alerted candidates to the fact that this was essentially a "write down" piece of working. However, many of the attempts that did get this far did not observe that the integrals in question were also valid for $n = 0$, and several found the answer by working backwards from I_1. Most candidates did not get this far in the question, so attempts at part **(ii)(c)** were seldom to be found. Very few of these considered the use of the *method of differences*, as specified in the question, which was intended to make the work more accessible, although acceptable alternatives often did appear. The final twist came with the informal understanding of convergence (advocated within the syllabus section headed Summation of series), but this required a clear grasp that $|\tanh x| < 1$ which was only seen by very few candidates. Nevertheless, those that did reach the end of the paper performed outstandingly well, and they had the opportunity to impress at a really advanced level; those scoring high marks have much cause to be proud of their efforts and abilities.

Answers: **(ii) (b)** $\frac{1}{2}\ln 3$; **(c)** $\ln 3$.