FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

Key messages

Candidates need to be more willing to give full explanations and justifications when these are required. They also need to consider the conciseness of their solutions to ensure that they do not run out of time to complete the paper.

General Comments

The standard of the work presented by the candidates was very good, with many achieving a three-figure total. In this paper, candidates were often being required to grasp the underlying ideas throughout a question, rather than being led through towards the solution. This meant that some of the shorter questions, in particular, proved to be quite tricky. **Questions 3, 7** and **10** were especially successful questions, while **Questions 11** and **12** were also popular sources of marks. **Question 13**, being the final question on the paper was often left incomplete, while **Question 6** was found difficult due to the fact that it required a firm grasp of the “big picture” in order for successful progress to be made. The most obvious general weakness amongst the candidature was the lack of enthusiasm for providing explanations and justifications; thus **Question 8**, a “theoretical” question on the group theory work, proved to be difficult for candidates. The other major hurdle to a complete attempt at all questions was the frequent deployment of very lengthy working in order to establish some very straightforward, routine tasks. For example, the bookwork at the start of **Question 13**, which often elicited up to two pages of working when a few lines should have been sufficient. The need for conciseness is paramount in these papers in order to allow candidates sufficient time to check their work.

Comments on specific questions

Question 1

This was a very gentle starter, and it was successfully attempted by almost all candidates. A lack of care when dealing with the details, such as in the handling of the constants involved in the “translated” standard integral or in dealing with the subtracted negative angle, resulted in some candidates not scoring full marks. A very small number of candidates lost the final mark by working in degrees.

Answer: $\frac{\sqrt{3}}{6}\pi$

Question 2

This was another straightforward question. However, none of the candidates gained all 4 marks on this question, as the majority were unaware of the requirement to justify the stated domain. A small number of candidates *did* realise that they should be addressing this issue, but they then did not do so satisfactorily. The series for $\ln(1 + x)$ is given in the List of Formulae, although some candidates ignored the prompt given in the question. They then had to amend this series for $\ln(1 - x)$, tidy up the difference of the two series with a bit of logarithmic work and point out that the answer was equal to the given series for $\text{arctanh}(x)$. However, there was also the one mark for noting, *explicitly*, that the domains for $\ln(1 + x)$ and $\ln(1 - x)$ needed to be “trimmed” slightly in order to make the stated result valid.

Question 3

Candidates clearly feel quite comfortable with the demands of the topic of rational functions, and marks scored on this question were high. However, very few scored all of the 9 marks available. As mentioned in the general comments, explanations and justifications were generally found insufficient, and the modest bit of work to show *convincingly* that the gradient of the given curve was indeed "always negative" was widely flawed, usually consisting of a comment to the effect that the numerator of the derivative consisted exclusively of "minus" terms.

Another small detail that might help future candidates is that it is expected that they will **note** "significant features" such as the coordinates of key points and the equations of asymptotes. On this question, where these were requested only to be shown, this was not so important and candidates were rewarded fully irrespective of whether they had shown the features on their sketch or noted them in writing alongside. However, it would be best if they were to be clear in what they wrote; for instance, rather than noting the asymptote "$y = 0$", many just drew it, while others stated it in "behavioural" forms such as "$y \to 0$" or "$y \to \frac{1}{x}$".

Question 4

This was a relatively routine vectors question, with part **(ii)** requiring little more than the use of a standard result. Those candidates who felt obliged to work out the shortest distance, by first determining the endpoints of the mutual perpendicular, made this question tougher than was intended.

Answers: **(i)** any (non-zero) multiple of $2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ **(ii)** $\sqrt{38}$

Question 5

Part **(i)** of this question was found to be very routine and, although part **(ii)** was also quite routine in principle, quite a few candidates seemed unable to approach the relevant use of *de Moivre's theorem* in this reversed way; that is, they seemed to know how to employ the work from part **(i)** to get, for example, $\sin 5\theta$ in terms of powers of $\sin\theta$, but were uncertain as to what was required in order to get a power of $\sin\theta$ in terms of the sines of multiple angles. These candidates apart, the biggest obstacle to a completely successful solution turned out to be the inability to keep the i's when using the result of part **(i)**; these often just disappeared altogether.

Answers: **(ii)** $A = \frac{1}{16}$, $B = -\frac{5}{16}$, $C = \frac{5}{8}$

Question 6

As mentioned in the introduction, this was found to be one of the more difficult questions on the paper. In some ways, this was a natural consequence of its formulation: those candidates who could not complete part **(i)** correctly were unable to produce the ensuing sketch in part **(ii)**. The result of not realising what was going on in both part **(i)** and part **(ii)** necessarily meant that the integral requested in part **(iii)** was most unlikely to be referenced to a region of the cartesian plane which would then have been very easily found. Many of those candidates who did derive the correct cartesian equation in part **(i)** still did not plot the graph of a simple parabola, adopting instead the approach of plotting individual points for the curve in its polar form; such sketches almost invariably curled back on themselves or formed a closed curve.

Answers: **(i)** $y = \frac{1}{2}(x^2 - 1)$ **(iii)** $\frac{4}{3}$

Question 7

This was the most successfully answered question on the paper. The most important part of the success that followed lay in the ability to factorise the sum of two cubes in part **(i)**; fortunately, almost all candidates found this straightforward and thus progressed through the question very comfortably.

Answers: **(i)** $(x + y)^3 - 3xy(x + y)$ **(ii) (a)** 19 **(b)** $19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$

Question 8

This group theory question was the least popular question on the paper, with around one in six candidates omitting it completely and, on average, it scored the poorest of all the questions. Lots of spurious reasons were given to justify the required results, with candidates clearly not sure how to write their justifications. In part **(i)(a)**, it was clear that many candidates did not realise the significance of the double-headed implication arrow, with many giving the result in only the one direction. Since one direction of the argument is trivial, this made the mark for it almost a giveaway, yet one which many candidates did not attempt. The simplest answer to part **(b)** would be the observation that $x\{G\}$ represents a row of the group table and is thus a permutation of the elements of G (by the *Latin Square* principle). The intended answer, however, was that candidates would realise, using the result of part **(i)(a)**, that each of the n xg_is were distinct elements of G and hence the full list of them merely formed a re-ordering of G's elements.

In part **(ii)**, despite the clear hint (and the near presence of the immediately preceding result), most efforts performed only one product of all of G's elements rather than the intended two. The other crucial, usually missing, component of a successful argument here was to observe the existence of an inverse element for each member of the group. Part **(iii)(a)** only required that candidates should note, by *Lagrange's theorem*, that elements can have an order which is a factor of n; and part **(b)** only required the observation that the result of part **(ii)** had relied on the *Abelian* nature of G.

Question 9

As with **Question 8**, many candidates omitted this question. For those candidates attempting the question, success revolved around how well they could write down the four matrices involved: the key phrase there being "write down". Although a bit of thought is needed to pin down the matrices for the two shears, many candidates spent their time working out from scratch those for the reflection and rotation, rather than using the List of Formulae and putting in the relevant numbers.

Most candidates who got this far figured out the correct order in which to multiply these four matrices; but many of these attempts foundered, however, when candidates kept the factors of $\frac{1}{\sqrt{2}}$ in each entry of the reflection and rotation matrices instead of taking them out as common factors (and multiplying them together to get $\frac{1}{2}$). All these obstacles meant that fully correct final matrices, and following descriptions, were rather few.

Answers: $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection in $y = x$

Question 10

This was the first of the four longer, structured, questions on the paper, and these questions brought most candidates a substantial proportion of their marks. The routine nature of part **(a)** was attempted by all candidates, and most of them successfully reached the end correctly. A few, however, left the complementary function (CF) in complex number form, which is not the accepted way of expressing the answer. Candidates should be encouraged to quote standard results, in this case, writing down the usual form of the CF when the auxiliary equation (AE) has complex roots, rather than deriving the CF by doing the background bookwork of substituting into the given differential equation solutions of the form Ae^{kx} in order to obtain the AE. For these second-order differential equations, it is expected that the AE will be written straight down, solved, and then the CF deduced immediately without any need to resort to any background justification.

Part **(ii)** was also reasonably routine and so was similarly well-received. The only common issue arose in the differentiation of the product term $y^2 \dfrac{dy}{dx}$, where there should have been a squared $\dfrac{dy}{dx}$ term which often appeared only singly.

Answers: **(a)** $k = -2$ $y = (A - 2x)\cos x + B\sin x$ **(b) (i)** 78 **(ii)** 2.013

Question 11

Part **(i)** was found straightforward by the majority of candidates, although many forgot to find two solutions to any of the possible resulting quadratics and one or two did not note that the positive value for p went with the negative value of q (and vice versa). Part **(ii)(a)** was intended to be little more than a simple bit of work on the relationships between coefficients and roots of the cubic equation; most candidates treated this part in this very simple way while others went about things in much more extended ways.

Part **(ii)(b)** was the only part of the question that quite a lot of candidates found difficult. Most of these candidates went about things in some unnecessarily lengthy ways, principally by solving the quadratic equation by setting up more relationships between roots and coefficients and then working with these, rather than just by solving the quadratic using the formula. It was hoped that candidates would reach this stage and appreciate the unity of the whole question, realising that the solution to part **(i)** was embedded in this final part of the question. However, very few candidates realised that this was the case: those that did finished the question quickly and efficiently, the rest usually struggled lengthily and with little ultimate success.

Answers: **(i)** $p = \pm 8$, $q = \mp 1$; **(ii) (a)** $A = 4 - 4\text{i}$, $B = 21 - 16\text{i}$, $C = 84$; **(b)** $1 - 4\text{i}$, $\frac{}{3} + \frac{}{3}\text{i}$.

Question 12

Induction questions are generally found to be difficult, and this was no exception. Nonetheless, despite the inherent difficulties associated with this kind of proof, very few candidates were not awarded the first four marks for obtaining the results for the cases $n = 1$ to 4 and almost all then spotted that the coefficient of x in the given linear multiple of e^{2x} was simply the corresponding power of 2. The constant term in this part of the required expression proved to be the big hindrance, and many gave up at this point. A few candidates carried on without it and gained a couple of the following marks. However, the very final mark, allocated to a completely satisfactory round-up of the inductive proof, was seldom gained, as most candidates resorted to a "QED" type of approach rather than give any explanation. The problem is that the "result for $n = k + 1$" needs to be ***shown*** to be of the right form, either by working out what it should be in advance or by writing it in the right form explicitly – in this case, $\dfrac{\text{d}^{n+1}y}{\text{d}x^{n+1}} = (2^{n+1}x + (n+1).2^{(n+1)-1})\,\text{e}^{2x}$.

Answers: **(i)** $(2x + 1)\text{e}^{2x}$, $(4x + 4)\text{e}^{2x}$, $(8x + 12)\text{e}^{2x}$, $(16x + 32)\text{e}^{2x}$. **(ii)** $(2^n x + n2^{n-1})\text{e}^{2x}$.

Question 13

After a straightforward opening, this was definitely a hard question, differentiating well between the candidates. It general, candidates did an unnecessary amount of work in proving both of the results in part **(i)**, though most attempts picked up 3 or 4 of the marks available. Part **(ii)** should have been attempted more readily by more candidates, especially as the given result can easily be established in either one of two ways: firstly by taking the direct hint in the question and looking at the single integral obtained when working with the given expression, $I_{n-1} - I_n$; alternatively by splitting up the terms of the given integral I_n appropriately and integrating by parts. Once again, many candidates did not look for any connection between the different parts of the question, and this second approach was facilitated by employing the identity of part **(i)(a)**.

The allocation of only 1 mark to part **(ii)(b)** should have alerted candidates to the fact that this was essentially a "write down" piece of working. However, many of the attempts that did get this far did not observe that the integrals in question were also valid for $n = 0$, and several found the answer by working backwards from I_1. Most candidates did not get this far in the question, so attempts at part **(ii)(c)** were seldom to be found. Very few of these considered the use of the *method of differences*, as specified in the question, which was intended to make the work more accessible, although acceptable alternatives often did appear. The final twist came with the informal understanding of convergence (advocated within the syllabus section headed Summation of series), but this required a clear grasp that $|\tanh x| < 1$ which was only seen by very few candidates. Nevertheless, those that did reach the end of the paper performed outstandingly well, and they had the opportunity to impress at a really advanced level; those scoring high marks have much cause to be proud of their efforts and abilities.

Answers: **(ii) (b)** $\frac{1}{2}\ln 3$; **(c)** $\ln 3$.

FURTHER MATHEMATICS

Paper 9795/02
Further Applications of Mathematics

Key Messages

In order to do well in this paper, candidates need to ensure that they have a sound working knowledge of both the mechanics and the probability sections of the syllabus.

General Comments

Candidates, as last year, responded well to this paper. There were many good scripts and also some of outstanding quality. In both sections of the paper there was a mixture of relatively straightforward questions and questions which offered a challenge to the most able. The disparity between performance on probability and mechanics was evident again this year, although some candidates performed better on the mechanics section, than on the probability section. The presentation of work was mostly of a good standard, with the majority of candidates presenting their work in an orderly and logical fashion, making it easy to follow their reasoning. Arithmetical accuracy was also good. Most candidates gave numerical answers to at least three significant figures, with only the occasional exception which was penalised in this respect. No instances of g being taken as 9.8 were observed this year. Algebraic manipulation, where necessary, was mostly of a good standard.

Most candidates seemed to have adequate time to complete the paper and to check their work. A few candidates could have improved their score by checking work more rigorously, small errors with signs, when standardising scores in probability questions, or sines and cosines being muddled in mechanics equations cost accuracy marks unnecessarily. Candidates seemed to have covered most areas of the syllabus and good work was seen in all questions on the paper. There was evidence of good preparation for this year's paper.

Comments on specific questions

Section A: Probability

Question 1

The first part of this question generally got candidates off to a good start. A small number of candidates used Po(25) rather than 'a suitable normal approximation', as they were asked to do in the question. This received less than full credit. Those who did not use a continuity correction with their normal approximation also lost credit, in both parts of the question. Most candidates had the right general idea in the second part of the question, but detail was often missing, such as a continuity correction, the sign of the standardised score, which meant that they were looking at the wrong tail of the distribution, or deciding that the answer was one less than the correct value.

Answers: **(i)** 0.0287 **(ii)** 34

Question 2

There were a pleasing number of correct answers to the calculation in the first part of the question, where it was expected that candidates would use the sampling distribution of the differences between means, $\overline{B} - \overline{G} \sim \mathrm{N}\left(-4, \frac{1465}{144}\right)$, although a number of candidates avoided this by using linear combinations of normal variables. The errors that occurred were mainly with using an incorrect variance, usually divisors being incorrect. There were few instances of variances being subtracted. The comments in part **(ii)** were frequently wide of the mark. Credit was awarded if candidates said that the samples were taken from underlying normal distributions.

Answer: **(i)** 0.105

Question 3

This year, candidates had a good idea of what they needed to find in the first part. This did not translate into full marks for the first part, since large numbers thought that $\mathrm{P}(X > 6)$ meant $1 - \mathrm{P}(X \leq 5)$ and a similar mistake was made with $\mathrm{P}(X > 3)$. Parts **(ii)** and **(iii)** were done correctly by most candidates, with only the occasional minor error being made.

Answers: **(i)** 0.324 **(iii)** 1.678

Question 4

Most candidates acquired the first two marks by finding means and standard deviations (or variances) for both villages *P* and *Q*. (The latter could be biased or unbiased at this stage.) A considerable number of candidates lost two marks because they did not calculate an unbiased estimate of the common population variance (or standard deviation). These candidates also lost the final mark for an inaccurate confidence interval but, otherwise, their work was followed through for a maximum of 5 of a possible 8 marks. For the comments in the second part, most realised that, when using the *t*-distribution, the samples had to be from normal distributions. The other comment required concerned the common population variance, which many had ignored in their calculation.

Answer: **(i)** (0.826, 2.10)

Question 5

Candidates scored reasonably well on this question. Most were able to get the correct value for *k* and the correct probability generating function, but relatively few knew that the modal value was the value with the greatest probability. In the second part, candidates demonstrated that they had a good understanding of the material. Those who lost marks mostly did so by errors in differentiating, or multiplying brackets. The latter was generally preferred to using the chain rule, which was somewhat surprising at this level.

Answers: **(i) (a)** $\frac{1}{10}$ **(b)** -1 **(ii) (a)** $\mathrm{G}_Y(t) = \frac{1}{100}\left(5t^{-1} + 3 + 2t^2\right)^2$ **(b)** $-\frac{1}{5}$, 2.58

Question 6

In the first part of the question, the majority of candidates earned a mark for F(x) = $\frac{x}{2}$. The second mark was frequently not earned, as the domain was either omitted completely, or included a '0 otherwise', indicating confusion with the probability density function. Most candidates managed to show, not always succinctly, the relationship between X and Y. They then attempted to find the cumulative distribution function for Y and hence the probability density function of Y, by differentiation. This approach led to errors with signs and domains. Only the occasional candidate opted for the more elegant approach of transforming $\int_0^2 \frac{1}{2}\,dx$ into $\int_0^{12} \frac{1}{4\sqrt{16-y}}\,dy$. A number of candidates were confused about how to find the median, using either the cdf or pdf of Y, but a considerable number found it successfully. Almost all knew how to find the expected value of Y, but their integration, usually using substitution, or parts, varied in accuracy considerably. The better candidates were generally successful though.

Answers: **(i)** $\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$ **(iii)** 7 **(iv)** $\frac{20}{3}$

Section B: Mechanics

Question 7

This question proved difficult for candidates, with many earning only two marks for finding the work done in raising the water 25 metres, either per second, or per minute. It was rare to find any candidate attempting to find the speed at which the water exited the pipe, consequently there were no attempts to find the kinetic energy gained, and thus attempts to calculate power had a term missing and were not awarded further marks.

Answer: **(i)** 13.5 kW

Question 8

This question was found more accessible, and the majority of candidates wrote down two equations, using $\dot{x}^2 = \omega^2\left(a^2 - x^2\right)$, which they were, successfully, able to solve simultaneously. In part **(ii)**, although candidates generally knew that $x = a\sin\omega t$ or $x = a\cos\omega t$ could be used to find the time, calculations became confused, inaccurate, or both.

Answers: **(i)** 3m, $\frac{\pi}{2}$sec or 1.57 sec **(ii)** $\frac{\pi}{8}$sec or 0.393 sec

Question 9

The first part required the use of the differential equation $-kv^3 = 2v\frac{dv}{dx}$ and most candidates realised this. There were some inaccuracies with the sign of the constant k. Most candidates made a good attempt to separate the variables and integrate. Since the final answer was given on the question paper, detailed and correct working, including either the calculation of a constant of integration, or use of limits, was required. In the second part the majority used the printed result and substituted $\frac{dx}{dt}$ for v, before separating variables and integrating. A somewhat smaller number started again, using $\frac{dv}{dt}$ for the acceleration. In both cases a considerable number of candidates found the time correctly. The errors that occurred mostly involved sign errors or, sometimes, careless arithmetic.

Answer: **(ii)** 3 sec

Question 10

In order to make progress in this question the direction in which *A* travels needs to be found. To achieve this, it is necessary to realise that *A* must travel perpendicular to the velocity of *A* relative to *B*. There were very few candidates who realised this. Those who did constructed a (3, 4, 5) triangle and found a relevant angle, which gave the bearing that *A* should take. A vector approach is possible, but it is necessary to use ${}_A\mathbf{v}_B.\mathbf{v}_A = 0$ and not, as all who tried vectors thought, ${}_A\mathbf{r}_B.\mathbf{v}_A = 0$. Taking $v_A = \begin{pmatrix} -16\sin\theta \\ 16\cos\theta \end{pmatrix}$ gives $\sin(\theta + 30°) = \frac{4}{5}$, from which θ and hence the bearing, can easily be found. For the second part, using the (3, 4, 5) triangle, it is easy to find the speed of *A* relative to *B* as 12 km h^{-1}. A further right-angled triangle calculation gives the distance that *A* must travel at this relative speed, and hence the time that it takes to reach the point when the ships are as close as possible. Those few adopting this approach were very often successful.

Answers: **(i)** 337° (nearest degree) **(ii)** 29.5 min

Question 11

Most candidates made a good attempt at conservation of energy in order to find the speed in part **(i)**. When trying to find the tension in the string, a term was often omitted from the equation. In the final part, weaker candidates found the radial acceleration only. Better candidates found the transverse acceleration as well and were able to deduce the resultant acceleration. Once again, a small amount of working was required, as the answer was displayed.

Answers: **(i)** 7.75 or $\sqrt{60}$ ms^{-1}, 70N

Question 12

A substantial number of candidates were able to do the first part of this question well. The conservation of momentum equation and Newton's law of restitution equation appeared in a variety of guises. Solving these two simultaneously gave the coefficient of restitution. The most able candidates were able to produce elegant and concise solutions here. The same two principles were needed for the second part, but the angles were less friendly and led to the quadratic equation $2t^2 - 5t - 3 = 0$, where $t = \tan\theta$. Weaker candidates found difficulty in reducing their two equations to a single equation in tan θ. A good proportion of candidates were able to do so. Solving this quadratic equation gave two values. Care needed to be exercised in explaining why one was rejected. A clear statement that the angle θ was acute, or equivalently $0° < \theta < 90°$, was required, again because the answer was given. There were relatively few who gained the final mark.

Answer: **(i)** $\frac{1}{9}$