

M13/5/FURMA/SP2/ENG/TZ0/XX



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International Baccalaureate®  
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**FURTHER MATHEMATICS  
STANDARD LEVEL  
PAPER 2**

Tuesday 21 May 2013 (morning)

2 hours

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**INSTRUCTIONS TO CANDIDATES**

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **Mathematics HL and Further Mathematics SL information booklet** is required for this paper.
- The maximum mark for this examination paper is [120 marks].

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 9]

The discrete random variable  $X$  follows the distribution  $\text{Geo}(p)$ .

(a) (i) Write down the mode of  $X$ .

(ii) Find the exact value of  $p$  if  $\text{Var}(X) = \frac{28}{9}$ . [3 marks]

Arthur tosses a biased coin each morning to decide whether to walk or cycle to school; he walks if the coin shows a head.

The probability of obtaining a head is 0.55.

(b) (i) Find the smallest value of  $n$  for which the probability of Arthur walking to school on the next  $n$  days is less than 0.01.

(ii) Find the probability that Arthur cycles to school for the third time on the last of eight successive days. [6 marks]

## 2. [Maximum mark: 27]

- (a) (i) Use partial fractions to show that the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent.
- (ii) Hence, by using the limit comparison test, determine whether the series  $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$  is convergent or divergent. [9 marks]
- (b) (i) Show that the improper integral  $\int_0^{\infty} \frac{1}{x^2+1} dx$  is convergent.
- (ii) Use the integral test to deduce that the series  $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$  is convergent, giving reasons why this test can be applied. [6 marks]
- (c) (i) Show that the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$  is convergent.
- (ii) If the sum of the above series is  $S$ , show that  $\frac{3}{5} < S < \frac{2}{3}$ . [6 marks]
- (d) For the series  $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$
- (i) determine the radius of convergence;
- (ii) determine the interval of convergence using your answers to (b) and (c). [6 marks]

## 3. [Maximum mark: 16]

A random variable  $X$  has probability density function  $f$  given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0 \text{ where } \lambda > 0 \\ 0, & \text{for } x < 0. \end{cases}$$

- (a) (i) Find an expression for  $P(X > a)$ , where  $a > 0$ .

A chicken crosses a road. It is known that cars pass the chicken's crossing route, with intervals between cars measured in seconds, according to the random variable  $X$ , with  $\lambda = 0.03$ . The chicken, which takes 10 seconds to cross the road, starts to cross just as one car passes.

- (ii) Find the probability that the chicken will reach the other side of the road before the next car arrives.

Later, the chicken crosses the road again just after a car has passed.

- (iii) Show that the probability that the chicken completes both crossings is greater than 0.5.

[6 marks]

- (b) A rifleman shoots at a circular target. The distance in centimetres from the centre of the target at which the bullet hits, can be modelled by  $X$  with  $\lambda = 0.4$ . The rifleman scores 10 points if  $X \leq 1$ , 5 points if  $1 < X \leq 5$ , 1 point if  $5 < X \leq 10$  and no points if  $X > 10$ .

- (i) Find the expected score when one bullet is fired at the target.

A second rifleman, whose shooting can also be modelled by  $X$ , wishes to find his value of  $\lambda$ .

- (ii) Given that his expected score is 6.5, find his value of  $\lambda$ .

[10 marks]

## 4. [Maximum mark: 24]

A group of people: Andrew, Betty, Chloe, David, Edward, Frank and Grace, has certain mutual friendships:

Andrew is friendly with Betty, Chloe, David and Edward;

Frank is friendly with Betty and Grace;

David, Chloe and Edward are friendly with one another.

- (a) (i) Draw the planar graph  $H$  that represents these mutual friendships.  
(ii) State how many faces this graph has. [3 marks]
- (b) Determine, giving reasons, whether  $H$  has  
(i) a Hamiltonian path;  
(ii) a Hamiltonian cycle;  
(iii) an Eulerian circuit;  
(iv) an Eulerian trail. [8 marks]
- (c) Verify Euler's formula for  $H$ . [2 marks]
- (d) State, giving a reason, whether or not  $H$  is bipartite. [2 marks]
- (e) Write down the adjacency matrix for  $H$ . [2 marks]

David wishes to send a message to Grace, in a sealed envelope, through mutual friends.

- (f) In how many different ways can this be achieved if the envelope is passed seven times and Grace only receives it once? [7 marks]

## 5. [Maximum mark: 21]

The set  $S$  consists of real numbers  $r$  of the form  $r = a + b\sqrt{2}$ , where  $a, b \in \mathbb{Z}$ .

The relation  $R$  is defined on  $S$  by  $r_1 R r_2$  if and only if  $a_1 \equiv a_2 \pmod{2}$  and  $b_1 \equiv b_2 \pmod{3}$ , where  $r_1 = a_1 + b_1\sqrt{2}$  and  $r_2 = a_2 + b_2\sqrt{2}$ .

- (a) Show that  $R$  is an equivalence relation. [7 marks]
- (b) Show, by giving a counter-example, that the statement  $r_1 R r_2 \Rightarrow r_1^2 R r_2^2$  is false. [3 marks]
- (c) Determine
- (i) the equivalence class  $E$  containing  $1 + \sqrt{2}$ ;
- (ii) the equivalence class  $F$  containing  $1 - \sqrt{2}$ . [3 marks]
- (d) Show that
- (i)  $(1 + \sqrt{2})^3 \in F$ ;
- (ii)  $(1 + \sqrt{2})^6 \in E$ . [4 marks]
- (e) Determine whether the set  $E$  forms a group under
- (i) the operation of addition;
- (ii) the operation of multiplication. [4 marks]

6. [Total mark: 23]

**Part A** [Maximum mark: 10]

- (a) Show that the opposite angles of a cyclic quadrilateral add up to  $180^\circ$ . [3 marks]
- (b) A quadrilateral ABCD is inscribed in a circle  $S$ . The four tangents to  $S$  at the vertices A, B, C and D form the edges of a quadrilateral EFGH. Given that EFGH is cyclic, show that AC and BD intersect at right angles. [7 marks]

**Part B** [Maximum mark: 13]

The circle  $C$  has centre O. The point Q is fixed in the plane of the circle and outside the circle. The point P is constrained to move on the circle.

- (a) Show that the locus of a point  $P'$ , which satisfies  $\vec{QP'} = k\vec{QP}$ , is a circle  $C'$ , where  $k$  is a constant and  $0 < k < 1$ . [6 marks]
- (b) Show that the two tangents to  $C$  from Q are also tangents to  $C'$ . [4 marks]

The circle  $C'$  cuts the line OQ at the points X and Y.

- (c) Show that  $QX \times QY = k^2 p$ , where  $p$  is the power of Q with respect to the circle  $C$ . [3 marks]