Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 1

Wednesday 20 May 2015 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

2215–7101

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0} (\csc x - \cot x)$.

2. [Maximum mark: 7]

(a) Find the general solution to the Diophantine equation $3x + 5y = 7$. [5]

(b) Find the values of x and y satisfying the equation for which x has the smallest positive integer value greater than 50. [2]

3. [Maximum mark: 11]

Consider the set $S = \{0, 1, 2, 3, 4, 5\}$ under the operation of addition modulo 6, denoted by $+_6$.

(a) Construct the Cayley table for $\{S, +_6\}$. [2]

(b) Show that $\{S, +_6\}$ forms an Abelian group. [5]

(c) State the order of each element. [2]

(d) Explain whether or not the group is cyclic. [2]

4. [Maximum mark: 10]

A simple graph G is represented by the following adjacency table.

	A	**B**	**C**	**D**	**E**	**F**
A	–	1	–	–	1	1
B	1	–	1	–	1	–
C	–	1	–	1	–	–
D	–	–	1	–	1	1
E	1	1	–	1	–	–
F	1	–	–	1	–	–

(a) Draw the simple graph G. [1]

(b) Explain why G does not contain an Eulerian circuit. [1]

(c) Show that G has a Hamiltonian cycle. [2]

(d) State whether or not G is planar, giving a reason for your answer. [2]

(e) State whether or not the simple graph G is bipartite, giving a reason for your answer. [2]

(f) Draw the complement G' of G. [2]

5. [Maximum mark: 9]

Jim is investigating the relationship between height and foot length in teenage boys.
A sample of 13 boys is taken and the height and foot length of each boy are measured.
The results are shown in the table.

Height x **cm**	129	135	156	146	155	152	139	166	148	179	157	152	160
Foot length y **cm**	25.8	25.9	29.7	28.6	29.0	29.1	25.3	29.9	26.1	30.0	27.6	27.2	28.0

You may assume that this is a random sample from a bivariate normal distribution.
Jim wishes to determine whether or not there is a positive association between height and foot length.

(a) Calculate the product moment correlation coefficient. [2]

(b) Find the p-value. [2]

(c) Interpret the p-value in the context of the question. [1]

(d) Find the equation of the regression line of y on x. [2]

(e) Estimate the foot length of a boy of height $170\,\text{cm}$. [2]

6. [Maximum mark: 9]

Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k^2}$.

7. [Maximum mark: 12]

(a) Sami is undertaking market research on packets of soap powder. He considers the brand “Gleam”. The weight of the contents of a randomly chosen packet of “Gleam” follows a normal distribution with mean 750 grams and standard deviation 20 grams. The weight of the packaging follows a different normal distribution with mean 40 grams and standard deviation 5 grams.

Find:

(i) the probability that a randomly chosen packet of “Gleam” has a **total** weight exceeding 780 grams.

(ii) the probability that the total weight of the **contents** of five randomly chosen packets of “Gleam” exceeds 3800 grams. [8]

(b) Sami now considers the brand “Bright”. The weight of the contents of a randomly chosen packet of “Bright” follow a normal distribution with mean 650 grams and standard deviation 16 grams. Find the probability that the **contents** of six randomly chosen packets of “Bright” weigh more than the **contents** of five randomly chosen packets of “Gleam”. [4]

8. [Maximum mark: 10]

(a) Differentiate the expression $x^2 \tan y$ with respect to x, where y is a function of x. [3]

(b) Hence solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x \sin 2y = x^3 \cos^2 y$ given that $y = 0$ when $x = 1$. Give your answer in the form $y = f(x)$. [7]

9. [Maximum mark: 10]

An integer N given in base r, can be expressed in base s in the form $N = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$ where $a_0, a_1, a_2, \ldots \in \{0, 1, 2, \ldots, s-1\}$.

(a) In base 5 an integer is written 1031. Express this integer in base 10. [2]

(b) Given that $N = 365$, $r = 10$ and $s = 7$ find the values of $a_0, a_1, a_2, \ldots$ [2]

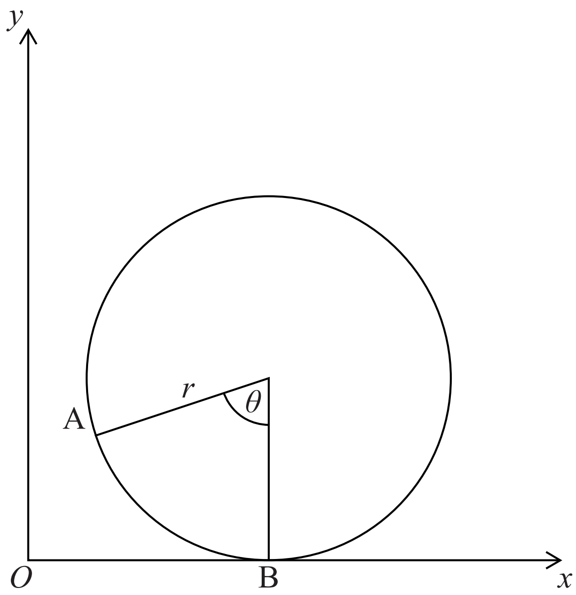
(c) (i) Given that $N = 899$, $r = 10$ and $s = 12$ find the values of $a_0, a_1, a_2, \ldots$

(ii) Hence write down the integer in base 12, which is equivalent to 899 in base 10. [3]

(d) Show that 121 is always a square number in any base greater than 2. [3]

10. [Maximum mark: 12]

A wheel of radius r rolls, without slipping, along a straight path with the plane of the wheel remaining vertical. A point A on the circumference of the wheel is initially at O. When the wheel is rolled, the radius rotates through an angle of θ and the point of contact is now at B, where the length of the arc AB is equal to the distance OB. This is shown in the following diagram.



(a) Find the coordinates of A in terms of r and θ. [3]

(b) As the wheel rolls, the point A traces out a curve. Show that the gradient of this curve is $\cot\left(\frac{1}{2}\theta\right)$. [6]

(c) Find the equation of the tangent to the curve when $\theta = \frac{\pi}{3}$. [3]

11. [Maximum mark: 7]

Prove that the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(x, y) = (2x + y, x + y)$ is a bijection.

12. [Maximum mark: 12]

A transformation T is a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$, represented by the matrix.

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 5 \\ -3 & 1 & 4 \\ 1 & 5 & 4 \end{pmatrix}$$

(a) (i) Find the row rank of M.

(ii) Hence or otherwise find the kernel of T. [8]

(b) (i) State the column rank of M.

(ii) Find the basis for the range of this transformation. [4]

13. [Maximum mark: 9]

(a) Two line segments $[\mathrm{AB}]$ and $[\mathrm{CD}]$ meet internally at the point Y. Given that $\mathrm{YA} \times \mathrm{YB} = \mathrm{YC} \times \mathrm{YD}$ show that A, B, C and D all lie on the circumference of a circle. [6]

(b) Explain why the result also holds if the line segments meet externally at Y. [3]

14. [Maximum mark: 9]

Sarah is the quality control manager for the Stronger Steel Corporation which makes steel sheets. The steel sheets should have a mean tensile strength of 430 MegaPascals (MPa). If the mean tensile strength drops to 400 MPa, then Sarah must recommend a change in composition. The tensile strength of these steel sheets follows a normal distribution with a standard deviation of 35 MPa. Sarah defines the following hypotheses

$$H_0 : \mu = 430$$

$$H_1 : \mu = 400$$

where μ denotes the mean tensile strength in MPa. She takes a random sample of n steel sheets and defines the critical region as $\bar{x} \le k$, where $\bar{x}$ denotes the mean tensile strength of the sample in MPa and k is a constant.

Given that the $P(\text{Type I Error}) = 0.0851$ and $P(\text{Type II Error}) = 0.115$, both correct to three significant figures, find the value of k and the value of n.

15. [Maximum mark: 13]

The relations ρ_1 and ρ_2 are defined on the Cartesian plane as follows

$$(x_1, y_1)\ \rho_1\ (x_2, y_2) \Leftrightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$$
$$(x_1, y_1)\ \rho_2\ (x_2, y_2) \Leftrightarrow \sqrt{x_1^2 + x_2^2} \le \sqrt{y_1^2 + y_2^2}\ .$$

(a) For ρ_1 and ρ_2 determine whether or not each is reflexive, symmetric and transitive. [11]

(b) For each of ρ_1 and ρ_2 which is an equivalence relation, describe the equivalence classes. [2]

16. [Maximum mark: 5]

A circle $x^2 + y^2 + dx + ey + c = 0$ and a straight line $lx + my + n = 0$ intersect. Find the general equation of a circle which passes through the points of intersection, justifying your answer.