Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

3 pages

8815 – 7208

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The function $f:\mathbb{R}\to\mathbb{R}$ is defined as $f: x \to \begin{cases} 1 & , x < 0 \\ 1 - x, & x \geq 0 \end{cases}$.

By considering limits, prove that f is

(a) continuous at $x = 0$; [2]

(b) not differentiable at $x = 0$. [3]

2. [Maximum mark: 10]

Let $f(x) = e^x \sin x$.

(a) Show that $f''(x) = 2(f'(x) - f(x))$. [4]

(b) By further differentiation of the result in part (a) , find the Maclaurin expansion of $f(x)$, as far as the term in x^5. [6]

3. [Maximum mark: 11]

(a) Prove by induction that $n! > 3^n$, for $n \geq 7$, $n \in \mathbb{Z}$. [5]

(b) Hence use the comparison test to prove that the series $\sum_{r=1}^{\infty} \frac{2^r}{r!}$ converges. [6]

4. [Maximum mark: 14]

Consider the function $f(x) = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$.

(a) Illustrate graphically the inequality, $\dfrac{1}{5}\displaystyle\sum_{r=1}^{5} f\left(\dfrac{r}{5}\right) < \int_0^1 f(x)\,\mathrm{d}x < \dfrac{1}{5}\sum_{r=0}^{4} f\left(\dfrac{r}{5}\right)$. [3]

(b) Use the inequality in part (a) to find a lower and upper bound for π. [5]

(c) Show that $\displaystyle\sum_{r=0}^{n-1} (-1)^r x^{2r} = \dfrac{1+(-1)^{n-1}x^{2n}}{1+x^2}$. [2]

(d) Hence show that $\pi = 4\left(\displaystyle\sum_{r=0}^{n-1} \dfrac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1 \dfrac{x^{2n}}{1+x^2}\,\mathrm{d}x\right)$. [4]

5. [Maximum mark: 20]

The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1, 0)$ and are defined by the differential equations $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

(a) Show that the tangent to the curve $y = f(x)$ at the point $(1, 0)$ is normal to the curve $y = g(x)$ at the point $(1, 0)$. [2]

(b) Find $g(x)$. [6]

(c) Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

(d) Explain why $y = f(x)$ cannot cross the isocline $x - y^2 = 0$, for $x > 1$. [3]

(e) (i) Sketch the isoclines $x - y^2 = -2, 0, 1$.

(ii) On the same set of axes, sketch the graph of f. [4]