

M13/5/FURMA/SP2/ENG/TZ0/XX



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**FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2**

Tuesday 21 May 2013 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **Mathematics HL and Further Mathematics SL information booklet** is required for this paper.
- The maximum mark for this examination paper is [120 marks].

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 9]

The discrete random variable X follows the distribution $\text{Geo}(p)$.

(a) (i) Write down the mode of X .

(ii) Find the exact value of p if $\text{Var}(X) = \frac{28}{9}$. [3 marks]

Arthur tosses a biased coin each morning to decide whether to walk or cycle to school; he walks if the coin shows a head.

The probability of obtaining a head is 0.55.

(b) (i) Find the smallest value of n for which the probability of Arthur walking to school on the next n days is less than 0.01.

(ii) Find the probability that Arthur cycles to school for the third time on the last of eight successive days. [6 marks]

2. [Maximum mark: 27]

- (a) (i) Use partial fractions to show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.
- (ii) Hence, by using the limit comparison test, determine whether the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ is convergent or divergent. [9 marks]
- (b) (i) Show that the improper integral $\int_0^{\infty} \frac{1}{x^2+1} dx$ is convergent.
- (ii) Use the integral test to deduce that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is convergent, giving reasons why this test can be applied. [6 marks]
- (c) (i) Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ is convergent.
- (ii) If the sum of the above series is S , show that $\frac{3}{5} < S < \frac{2}{3}$. [6 marks]
- (d) For the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$
- (i) determine the radius of convergence;
- (ii) determine the interval of convergence using your answers to (b) and (c). [6 marks]

3. [Maximum mark: 16]

A random variable X has probability density function f given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0 \text{ where } \lambda > 0 \\ 0, & \text{for } x < 0. \end{cases}$$

- (a) (i) Find an expression for $P(X > a)$, where $a > 0$.

A chicken crosses a road. It is known that cars pass the chicken's crossing route, with intervals between cars measured in seconds, according to the random variable X , with $\lambda = 0.03$. The chicken, which takes 10 seconds to cross the road, starts to cross just as one car passes.

- (ii) Find the probability that the chicken will reach the other side of the road before the next car arrives.

Later, the chicken crosses the road again just after a car has passed.

- (iii) Show that the probability that the chicken completes both crossings is greater than 0.5.

[6 marks]

- (b) A rifleman shoots at a circular target. The distance in centimetres from the centre of the target at which the bullet hits, can be modelled by X with $\lambda = 0.4$. The rifleman scores 10 points if $X \leq 1$, 5 points if $1 < X \leq 5$, 1 point if $5 < X \leq 10$ and no points if $X > 10$.

- (i) Find the expected score when one bullet is fired at the target.

A second rifleman, whose shooting can also be modelled by X , wishes to find his value of λ .

- (ii) Given that his expected score is 6.5, find his value of λ .

[10 marks]

4. [Maximum mark: 24]

A group of people: Andrew, Betty, Chloe, David, Edward, Frank and Grace, has certain mutual friendships:

Andrew is friendly with Betty, Chloe, David and Edward;

Frank is friendly with Betty and Grace;

David, Chloe and Edward are friendly with one another.

- (a) (i) Draw the planar graph H that represents these mutual friendships.
(ii) State how many faces this graph has. [3 marks]
- (b) Determine, giving reasons, whether H has
(i) a Hamiltonian path;
(ii) a Hamiltonian cycle;
(iii) an Eulerian circuit;
(iv) an Eulerian trail. [8 marks]
- (c) Verify Euler's formula for H . [2 marks]
- (d) State, giving a reason, whether or not H is bipartite. [2 marks]
- (e) Write down the adjacency matrix for H . [2 marks]

David wishes to send a message to Grace, in a sealed envelope, through mutual friends.

- (f) In how many different ways can this be achieved if the envelope is passed seven times and Grace only receives it once? [7 marks]

5. [Maximum mark: 21]

The set S consists of real numbers r of the form $r = a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

The relation R is defined on S by $r_1 R r_2$ if and only if $a_1 \equiv a_2 \pmod{2}$ and $b_1 \equiv b_2 \pmod{3}$, where $r_1 = a_1 + b_1\sqrt{2}$ and $r_2 = a_2 + b_2\sqrt{2}$.

- (a) Show that R is an equivalence relation. [7 marks]
- (b) Show, by giving a counter-example, that the statement $r_1 R r_2 \Rightarrow r_1^2 R r_2^2$ is false. [3 marks]
- (c) Determine
- (i) the equivalence class E containing $1 + \sqrt{2}$;
- (ii) the equivalence class F containing $1 - \sqrt{2}$. [3 marks]
- (d) Show that
- (i) $(1 + \sqrt{2})^3 \in F$;
- (ii) $(1 + \sqrt{2})^6 \in E$. [4 marks]
- (e) Determine whether the set E forms a group under
- (i) the operation of addition;
- (ii) the operation of multiplication. [4 marks]

6. [Total mark: 23]

Part A [Maximum mark: 10]

- (a) Show that the opposite angles of a cyclic quadrilateral add up to 180° . [3 marks]
- (b) A quadrilateral ABCD is inscribed in a circle S . The four tangents to S at the vertices A, B, C and D form the edges of a quadrilateral EFGH. Given that EFGH is cyclic, show that AC and BD intersect at right angles. [7 marks]

Part B [Maximum mark: 13]

The circle C has centre O. The point Q is fixed in the plane of the circle and outside the circle. The point P is constrained to move on the circle.

- (a) Show that the locus of a point P' , which satisfies $\vec{QP'} = k\vec{QP}$, is a circle C' , where k is a constant and $0 < k < 1$. [6 marks]
- (b) Show that the two tangents to C from Q are also tangents to C' . [4 marks]

The circle C' cuts the line OQ at the points X and Y.

- (c) Show that $QX \times QY = k^2 p$, where p is the power of Q with respect to the circle C . [3 marks]