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**Further mathematics**  
**Higher level**  
**Paper 1**

Thursday 23 May 2019 (afternoon)

2 hours 30 minutes

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

The graph  $G$  with vertices  $A, B, C, D$  and  $E$  has the weights shown in the following table.

	A	B	C	D	E
A	–	8	11	17	12
B	8	–	14	9	13
C	11	14	–	16	10
D	17	9	16	–	15
E	12	13	10	15	–

- (a) Justifying your answer, explain whether or not  $G$  contains an Eulerian circuit. [2]
- (b) Prove that  $G$  cannot be drawn as a planar graph. [3]
- (c) Starting at  $A$ , use the nearest-neighbour algorithm to find an upper bound for the travelling salesman problem for  $G$ . [3]
- (d) By deleting vertex  $A$ , use the deleted vertex algorithm to find a lower bound for this travelling salesman problem. [4]

2. [Maximum mark: 9]

The function  $f$  is defined for  $x \geq 0$  by  $f(x) = \ln(2e^x - 1)$ .

- (a) Determine the Maclaurin series for  $f(x)$  as far as the term in  $x^3$ . [7]

- (b) Hence determine the value of

$$\lim_{x \rightarrow 0} \frac{f(x) - 2x}{x^2}. \quad [2]$$

3. [Maximum mark: 12]

(a) The matrix  $A$  is given by  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

(i) Show that the eigenvalues of  $A$  are real if  $(a - d)^2 + 4bc \geq 0$ .

(ii) Deduce that the eigenvalues are real if  $A$  is symmetric. [6]

(b) The matrix  $B$  is given by  $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ .

(i) Determine the eigenvalues of  $B$ .

(ii) Determine the corresponding eigenvectors. [6]

4. [Maximum mark: 8]

The positive integer  $N$  is given by 1321 when expressed in base  $b$  and 521 when expressed in base  $b + 2$ .

(a) Determine the value of  $b$ . [4]

(b) Express  $N$

(i) in base 10;

(ii) in base 16. [4]

5. [Maximum mark: 9]

Consider the differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x \quad \text{where } 0 \leq x < \frac{\pi}{2}$$

Given that  $y = 2$  when  $x = 0$ , solve the differential equation giving your answer in the form  $y = f(x)$ . [9]

6. [Maximum mark: 8]

The group  $\{G, *\}$  has the following Cayley table.

*	0	1	2	3	4	5
0	4	5	0	1	2	3
1	5	2	1	4	3	0
2	0	1	2	3	4	5
3	1	4	3	0	5	2
4	2	3	4	5	0	1
5	3	0	5	2	1	4

- (a) Determine the order of each of the elements of  $\{G, *\}$ . [4]
- (b) Hence find the subgroup  $S_2$  of order 2 and the subgroup  $S_3$  of order 3. [2]
- (c) Write down the coset with respect to  $S_2$  of each element of  $\{G, *\}$  not included in  $S_2$ . [2]

7. [Maximum mark: 12]

- (a) Determine the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$ . [5]
- (b) (i) Use l'Hôpital's rule to determine the value of  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$ .
- (ii) Use the limit comparison test together with an appropriate series to determine whether the series  $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$  is convergent or divergent. [7]

8. [Maximum mark: 7]

The line AD is a median of the acute-angled triangle ABC and E is the midpoint of AD. The line BE meets AC at the point F.

- (a) Draw a diagram to illustrate this situation. [1]
- (b) Determine the value of the ratio  $\frac{CF}{AF}$ . [4]
- (c) The line CE meets AB at the point G. Giving a reason, write down the value of the ratio  $\frac{BG}{AG}$ . [2]

9. [Maximum mark: 12]

Consider the system of equations

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 4 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \mu \end{bmatrix}$$

where the matrix  $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 4 & 1 & 6 \end{bmatrix}$  is singular and  $\mu$  is a constant.

- (a) Determine the value of  $\mu$  for which the equations are consistent. [4]
- (b) For this value of  $\mu$
- (i) solve the system of equations;
- (ii) find the values of  $x, y$  and  $z$  which minimize  $x^2 + y^2 + z^2$  and interpret your result geometrically. [8]

10. [Maximum mark: 10]

The continuous random variable  $X$  has cumulative distribution function  $F$  given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{\pi} \arctan x, & 0 \leq x < \infty \end{cases} .$$

- (a) (i) Sketch the graph of  $F(x)$  for  $x \geq 0$ .
- (ii) Explain how it can be deduced from the graph of  $F(x)$  that the mode of  $X$  is zero.
- (iii) Determine the median of  $X$ . [5]
- (b) It is often stated that for certain probability distributions, the following approximation is true:

$$\text{Median} - \text{Mode} \approx 2(\text{Mean} - \text{Median}).$$

Explain why this approximation is not valid for the probability distribution defined above. [5]

11. [Maximum mark: 10]

- (a) (i) Show that the set  $S_1$  of three-dimensional vectors given by

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} \right\}$$

is a basis for three-dimensional vectors.

- (ii) Express the vector  $\begin{bmatrix} 9 \\ 17 \\ 3 \end{bmatrix}$  in terms of  $S_1$ . [5]

- (b) (i) Show that the set  $S_2$  of three-dimensional vectors given by

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix} \right\}$$

is not a basis for three-dimensional vectors.

- (ii) State the dimension of the subspace spanned by  $S_2$ .

- (iii) Determine whether or not the vector  $\begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$  belongs to this subspace. [5]

12. [Maximum mark: 9]

The relation  $R$  is defined on  $\mathbb{Z}^+$  such that  $xRy$  if and only if  $x^2 - y^2 \equiv 0 \pmod{N}$  where  $N \geq 3$  is a positive integer.

- (a) Show that  $R$  is an equivalence relation for all values of  $N$ . [6]

- (b) Show that  $N - 1$  and  $N + 1$  are in the same equivalence class as 1. [3]

13. [Maximum mark: 11]

The function  $f: \mathbf{M} \rightarrow \mathbf{M}$  where  $\mathbf{M}$  is the set of  $2 \times 2$  matrices, is given by  $f(\mathbf{X}) = \mathbf{A}\mathbf{X}$  where  $\mathbf{A}$  is a  $2 \times 2$  matrix.

(a) Given that  $\mathbf{A}$  is non-singular, prove that  $f$  is a bijection. [7]

It is now given that  $\mathbf{A}$  is singular.

(b) By considering appropriate determinants, prove that  $f$  is not a bijection. [4]

14. [Maximum mark: 12]

The Poisson random variable  $X$  with mean  $m$  has probability function

$$P(X = x) = \frac{e^{-m} m^x}{x!}, x \in \mathbb{N}.$$

(a) Show that the probability generating function of  $X$  is given by

$$G_x(t) = e^{m(t-1)}. \quad [3]$$

(b) A random sample  $X_1, X_2, X_3$  is taken from the distribution of  $X$ . The random variable  $Y$  is defined by  $Y = X_1 + 2X_2 + 3X_3$ .

(i) Show that the probability generating function of  $Y$  is given by  $G_y(t) = e^{-3m} e^{m(t+t^2+t^3)}$ .

(ii) By considering the series expansion of  $e^{m(t+t^2+t^3)}$ , determine an expression in terms of  $m$  for  $P(Y = 4)$ . [9]



**15.** [Maximum mark: 9]

An ellipse  $E$  has equation  $x^2 + 2y^2 = 2$ . The point  $P$  has coordinates  $(x_1, y_1)$  and is external to the ellipse.

(a) Write down the equation of the line  $L$  with gradient  $m$  passing through the point  $P$ . [1]

(b) Show that the  $x$  coordinates of the points of intersection of the line  $L$  and the ellipse  $E$  are given by the roots of the quadratic equation

$$x^2(1 + 2m^2) + 4mx(y_1 - mx_1) + 2y_1^2 + 2m^2x_1^2 - 4mx_1y_1 - 2 = 0. \quad [3]$$

(c) Show that the condition for the line  $L$  to be a tangent to  $E$  is given by

$$m^2(x_1^2 - 2) - 2mx_1y_1 + y_1^2 - 1 = 0. \quad [3]$$

(d) Hence show that the equation of the locus of points from which the two tangents to  $E$  are perpendicular is  $x^2 + y^2 = 3$ . [2]

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