M19/5/FURMA/HP2/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2019

Further mathematics

Higher level

Paper 2

23 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) attempt to calculate differences (either way round) ***(M1)***
$6, -13, 1, -5, -1, 2, -5, -1, -7, -2$ ***(A1)***
attempt to carry out a *t*-test on the differences ***(M1)***
$\text{DF} = 9$, $t = \pm 1.49$ ***A1A1***
p-value $= 0.170$ ***A1***

(ii) insufficient evidence to say that there is a difference in mean marks on the two papers ***R1***
(since p-value is greater than 5% and 10%)

[7 marks]

(b) (i) $H_0 : \rho = 0$, $H_1 : \rho > 0$ ***A1***

(ii) $r = 0.920$ ***A2***

(iii) p-value $= 8.01 \times 10^{-5}$ ***A1***
(very strong) evidence that there is a positive association between the marks ***R1***

[5 marks]

Total [12 marks]

2. (a) since $(1,0) \to (\cos\theta, \sin\theta)$ and $(0,1) \to (-\sin\theta, \cos\theta)$, ***M1A1***

required matrix $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ***AG***

[2 marks]

(b) (i) **METHOD 1**

let $(x, y) \to (X, Y)$ under the anticlockwise rotation so that

$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ ***(M1)***

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$ ***A1***

the equation of C_2 is

$\frac{1}{2}(X+Y)^2 + \frac{1}{2}(-X+Y)^2 - \frac{6}{2}(X+Y)(-X+Y) - 4 = 0$ ***M1A1***

$\frac{1}{2}X^2 + \frac{1}{2}Y^2 + XY + \frac{1}{2}X^2 + \frac{1}{2}Y^2 - XY - 3Y^2 + 3X^2 - 4 = 0$ ***A1***

$4X^2 - 2Y^2 - 4 = 0$ ***A1***

$x^2 - \frac{y^2}{2} = 1$ ***AG***

continued…

Question 2 continued

METHOD 2

C_1 can be written as $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - 4 = 0$

$\det\left(\begin{bmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 4$ ***(M1)***

$\lambda_1 = -2$:

$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = x$ and so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector ***A1***

$\lambda_2 = 4$:

$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = -x$ and so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector ***A1***

normalised eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boldsymbol{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ ***A1***

$\boldsymbol{R}^T\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\boldsymbol{R} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ ***M1***

C_2 can be written as $\begin{bmatrix} X & Y \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} - 4 = 0$

$4X^2 - 2Y^2 - 4 = 0$ ***A1***

$x^2 - \frac{y^2}{2} = 1$ ***AG***

continued…

Question 2 continued

METHOD 3

let $(x, y) \to (X, Y)$ under the anticlockwise rotation so that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ ***(M1)***

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$ ***A1***

the equation of C_2 is

$$(X\cos\theta + Y\sin\theta)^2 + (-X\sin\theta + Y\cos\theta)^2$$
$$-6(X\cos\theta + Y\sin\theta)(-X\sin\theta + Y\cos\theta) - 4 = 0$$ ***M1A1***

$$X^2(\cos^2\theta + \sin^2\theta + 6\sin\theta\cos\theta)$$
$$+XY(2\sin\theta\cos\theta - 2\sin\theta\cos\theta - 6\cos^2\theta + 6\sin^2\theta)$$ ***A1***
$$+Y^2(\sin^2\theta + \cos^2\theta - 6\sin\theta\cos\theta) - 4 = 0$$

when $\theta = 45°$:

$$X^2(1 + 6\sin\theta\cos\theta) + Y^2(1 - 6\sin\theta\cos\theta) - 4 = 0$$
$$4X^2 - 2Y^2 - 4 = 0$$ ***A1***
$$x^2 - \frac{y^2}{2} = 1$$ ***AG***

(ii) hyperbola ***A1***

[7 marks]

(c) (i) in standard notation, $a^2 = 1$ and $b^2 = 2$ ***(A1)***

using $e = \frac{\sqrt{a^2 + b^2}}{a}$ (or equivalent) ***(M1)***

$e = \sqrt{3}$ ***A1***

Note: Award ***FT*** for their value of *e*.

(ii) using $\left(\pm\sqrt{a^2 + b^2}, 0\right)$ or $(\pm ae, 0)$ ***(M1)***

$= \left(\pm\sqrt{3}, 0\right)$ ***A1***

(iii) using $x = \pm\frac{a}{e}$ ***(M1)***

$x = \pm\frac{1}{\sqrt{3}}$ ***A1***

[7 marks]

continued…

Question 2 continued

(d) (i) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \pm\sqrt{3} \\ 0 \end{bmatrix}$ ***(M1)***

$= \left(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}\right)$ and $\left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ ***A1***

(ii) $\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = \pm\frac{1}{\sqrt{3}}$ $\left(y = x \pm \frac{\sqrt{6}}{3}\right)$ ***(M1)A1***

[4 marks]

Total [20 marks]

3. (a) **METHOD 1**

$$\boldsymbol{M}^2 = \begin{bmatrix} 11 & 10 & 6 \\ 8 & 10 & 8 \\ 13 & 10 & 8 \end{bmatrix};\ \boldsymbol{M}^3 = \begin{bmatrix} 53 & 50 & 38 \\ 54 & 50 & 34 \\ 59 & 60 & 44 \end{bmatrix}$$ ***(A1)(A1)***

let $\begin{bmatrix} 53 & 50 & 38 \\ 54 & 50 & 34 \\ 59 & 60 & 44 \end{bmatrix} = a\begin{bmatrix} 11 & 10 & 6 \\ 8 & 10 & 8 \\ 13 & 10 & 8 \end{bmatrix} + b\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ***(M1)***

then, for example

$11a + b + c = 53$
$10a + 2b = 50$ ***M1A1***
$6a + 2b = 38$

the solution is $a = 3, b = 10, c = 10$ ***(M1)A1***

($\boldsymbol{M}^3 = 3\boldsymbol{M}^2 + 10\boldsymbol{M} + 10\boldsymbol{I}$)

[7 marks]

METHOD 2

$\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$ ***(M1)***

$\Rightarrow (1-\lambda)\left((1-\lambda)^2 - 3\right) - 2\left(3(1-\lambda) - 2\right) + 2\left(9 - 2(1-\lambda)\right) = 0$ ***M1A1***

$-\lambda^3 + 3\lambda^2 + 10\lambda + 10 = 0$ ***M1A1***

applying the Cayley – Hamilton theorem ***(M1)***

$\boldsymbol{M}^3 = 3\boldsymbol{M}^2 + 10\boldsymbol{M} + 10\boldsymbol{I}$ and so $a = 3,\ b = 10,\ c = 10$ ***A1***

[7 marks]

continued…

Question 3 continued

(b) $\boldsymbol{M}^4 = 3\boldsymbol{M}^3 + 10\boldsymbol{M}^2 + 10\boldsymbol{M}$ ***M1***

$= 3\left(3\boldsymbol{M}^2 + 10\boldsymbol{M} + 10\boldsymbol{I}\right) + 10\boldsymbol{M}^2 + 10\boldsymbol{M}$ ***M1***

$= 19\boldsymbol{M}^2 + 40\boldsymbol{M} + 30\boldsymbol{I}$ ***AG***

[2 marks]

(c) the statement is true for $n = 3$ as shown in part (a) ***A1***

assume true for $n = k$, *ie* $\boldsymbol{M}^k = p\boldsymbol{M}^2 + q\boldsymbol{M} + r\boldsymbol{I}$ ***M1***

Note: Subsequent marks after this ***M1*** are independent and can be awarded.

$\boldsymbol{M}^{k+1} = p\boldsymbol{M}^3 + q\boldsymbol{M}^2 + r\boldsymbol{M}$ ***M1***

$= p\left(3\boldsymbol{M}^2 + 10\boldsymbol{M} + 10\boldsymbol{I}\right) + q\boldsymbol{M}^2 + r\boldsymbol{M}$ ***M1***

$= (3p + q)\boldsymbol{M}^2 + (10p + r)\boldsymbol{M} + 10p\boldsymbol{I}$ ***A1***

hence true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 3$,
the statement is proved by induction ***R1***

Note: Award ***R1*** provided at least four of the previous marks are gained.

[6 marks]

(d) $\boldsymbol{M}^2 = 3\boldsymbol{M} + 10\boldsymbol{I} + 10\boldsymbol{M}^{-1}$ ***M1***

$\boldsymbol{M}^{-1} = \frac{1}{10}\left(\boldsymbol{M}^2 - 3\boldsymbol{M} - 10\boldsymbol{I}\right)$ ***A1***

[2 marks]

Total [17 marks]

4. (a) (i) Fermat's little theorem states that
$a^{p-1} \equiv 1(\text{mod } p)$ where p is prime and p does not divide a ***A1***

(ii) we know that $7^{22} = 1 + 23N$ where $N \in \mathbb{Z}^+$ ***(M1)***
$7^{22n} = (1+23N)^n$ ***(A1)***
$= 1 + n \times 23N + \dfrac{n(n-1)}{2} \times (23N)^2 + \ldots + (23N)^n$ ***A1***
$\equiv 1(\text{mod } 23)$ ***AG***

(iii) $7^{2209} \equiv 7^{100\times 22+9} (\text{mod } 23)$ ***(M1)***
$\equiv 7^9 (\text{mod } 23)$ ***(A1)***
$\equiv 15(\text{mod } 23)$ ***A1***

[7 marks]

(b) (i) producing a table or using trial and error ***(M1)***

a	$7^a (\text{mod} 22)$
1	7
2	5
3	13
4	3
5	21
6	15
7	17
8	9
9	19
10	1

EITHER
$7^{10} \equiv 1(\text{mod } 22)$ ***(A1)***
OR
$7^5 \equiv -1(\text{mod } 22) \Rightarrow 7^{10} \equiv (-1)^2 (\text{mod } 22)$ ***(A1)***
$a = 10$ ***A1***

(ii) $b = 8$ ***A1***
EITHER
possible values 8, 18, 28, … ***A1***
OR
general solution is $b = 8 + 10N$ ***A1***

[5 marks]

Total [12 marks]

5. (a) (i)

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	**5**
3	3	6	2	5	**1**	**4**
4	4	1	**5**	**2**	**6**	**3**
5	5	**3**	**1**	**6**	**4**	**2**
6	6	**5**	**4**	**3**	**2**	**1**

A3

Note: Award ***A3*** for all correct, ***A2*** for at most 2 errors, ***A1*** for at most 4 errors.

(ii) closure: all elements in the Cayley table belong to G ***A1***
identity: the identity is 1 ***A1***
all elements have an inverse, 1 and 6 are self-inverse
3, 5 and 2, 4 are inverse pairs ***A1***
multiplication is associative ***A1***
the four group axioms are satisfied and so $\{G, \times_7\}$ is a group ***AG***

(iii) attempt to find a generator of order 6 ***(M1)***
G is cyclic because 3 (or 5) generates the group ***A1***

[9 marks]

(b) $A = \{a, a^2, a^3, \ldots, a^n (= e)\}$ and $B = \{b, b^2, b^3, \ldots, b^n (= e)\}$ ***(A1)***

consider the bijection f from A to B given by $a^p \to b^p$, $1 \le p \le n$

then $f(a^r a^s) = f(a^{r+s})$ ***A1***

$= b^{r+s}$ (where $r+s$ is evaluated mod n) ***A1***

and $f(a^r) f(a^s) = b^r b^s = b^{r+s}$ ***M1A1***

hence $f(a^r a^s) = f(a^r) f(a^s)$ (which is the condition for an isomorphism) ***R1***

so A and B are isomorphic ***AG***

[6 marks]

(c) $1 \to 0$ in both bijections (identity match) ***R1***
$6 \to 3$ in both bijections (due to order) ***R1***
5 or $3 \to 1$ or 5 (generators match and injectivity) ***R1R1***
$2 \to 2$, $4 \to 4$ and $2 \to 4$, $4 \to 2$ (due to order or what is left to have a bijection)
A1A1

[6 marks]

Total [21 marks]

6. (a) put $u = x^3 \Rightarrow \mathrm{d}u = 3x^2\mathrm{d}x$ ***(A1)***

$\int \frac{x^2\mathrm{d}x}{x^6+1} = \frac{1}{3}\int \frac{\mathrm{d}u}{u^2+1}$ ***M1***

$= \frac{1}{3}\arctan u$

$= \frac{1}{3}\arctan(x^3) + \text{constant}$ ***AG***

[2 marks]

(b) (i) the diagram shows the curve and upper and lower rectangles

$\int_n^\infty \frac{x^2\mathrm{d}x}{x^6+1}$ represents the area under the curve between n and ∞ ***R1***

it follows that it is less than the sum of the upper rectangles
but greater than the sum of the lower rectangles, ***R1***

ie $\sum_{r=n+1}^{\infty} \frac{r^2}{r^6+1} < \int_n^\infty \frac{x^2\mathrm{d}x}{x^6+1} < \sum_{r=n}^{\infty} \frac{r^2}{r^6+1}$ ***AG***

(ii) $\int_n^\infty \frac{x^2\mathrm{d}x}{x^6+1} = \frac{1}{3}\left[\arctan(x^3)\right]_n^\infty$ ***M1***

$= \frac{\pi}{6} - \frac{1}{3}\arctan(n^3)$ ***A1***

consider

$S = \sum_{r=1}^{\infty} \frac{r^2}{r^6+1} = \sum_{r=1}^{n} \frac{r^2}{r^6+1} + \sum_{r=n+1}^{\infty} \frac{r^2}{r^6+1}$ ***M1A1***

$< \sum_{r=1}^{n} \frac{r^2}{r^6+1} + \frac{\pi}{6} - \frac{1}{3}\arctan(n^3)$, giving an upper bound for S ***AG***

(iii) **EITHER**

now consider

$S = \sum_{r=1}^{\infty} \frac{r^2}{r^6+1} = \sum_{r=1}^{n-1} \frac{r^2}{r^6+1} + \sum_{r=n}^{\infty} \frac{r^2}{r^6+1}$ ***(M1)(A1)***

$> \sum_{r=1}^{n-1} \frac{r^2}{r^6+1} + \frac{\pi}{6} - \frac{1}{3}\arctan(n^3)$, giving a lower bound for S ***A1***

OR

now consider

$S = \sum_{r=1}^{\infty} \frac{r^2}{r^6+1} = \sum_{r=1}^{n} \frac{r^2}{r^6+1} + \sum_{r=n+1}^{\infty} \frac{r^2}{r^6+1}$ ***(M1)(A1)***

$> \sum_{r=1}^{n} \frac{r^2}{r^6+1} + \frac{\pi}{6} - \frac{1}{3}\arctan\left((n+1)^3\right)$, giving a lower bound for S ***A1***

continued…

Question 6 continued

(iv) putting $n = 8$, ***(M1)***
lower bound $= 0.5812...$ ***(A1)***
upper bound $= 0.5814...$ ***(A1)***
therefore $S = 0.581$ (correct to 3 decimal places) ***A1***

[13 marks]

Total [15 marks]

7. (a) $0 \le 2\theta \le 1 \Rightarrow 0 \le \theta \le \frac{1}{2}$ and $0 \le 1 - 3\theta \le 1 \Rightarrow 0 \le \theta \le \frac{1}{3}$ ***(M1)***

$0 \le \theta \le \frac{1}{3}$ ***A1***

Note: Accept strong inequalities.

[2 marks]

(b)

θ

x^n	0	1^n	2^n
$\mathrm{P}(X = x)$		2θ	$1-3\theta$

(M1)

$\mathrm{E}(X^n) = 1^n \times 2\theta + 2^n(1 - 3\theta)$ ***A1***

$= 2^n + 2\theta - 2^n \times 3\theta$

$= 2^n + 2\theta(1 - 3 \times 2^{n-1})$ ***AG***

[2 marks]

(c) $\mathrm{E}(X) = 2 - 4\theta$ ***A1***

$\mathrm{E}(T_1) = \frac{1}{2} - \frac{1}{4n} \times n(2 - 4\theta)$ ***M1A1***

$= \theta$, therefore T_1 is an unbiased estimator for θ ***AG***

[3 marks]

(d) $\mathrm{E}(X^2) = 4 - 10\theta$ ***A1***

$\mathrm{E}(T_2) = \frac{2}{5} - \frac{1}{10n} \times n(4 - 10\theta)$ ***A1***

$= \theta$, therefore T_2 is an unbiased estimator for θ ***AG***

[2 marks]

(e) (i) use of $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$ ***(M1)***

$= 4 - 10\theta - (2 - 4\theta)^2$ ***A1***

$= 6\theta - 16\theta^2$ ***AG***

continued…

Question 7 continued

(ii) $\text{Var}\left(X^2\right)=\text{E}(X^4)-\left(\text{E}(X^2)\right)^2$ ***(M1)***

$\text{E}(X^4)=2\theta+16(1-3\theta)\ \ (=16-46\theta)$ ***(A1)***

$\text{Var}\left(X^2\right)=2\theta+16(1-3\theta)-(4-10\theta)^2$ ***M1***

$=34\theta-100\theta^2$ ***A1***

$(a=34, b=100)$

[6 marks]

(f) $\text{Var}(T_1)=\dfrac{1}{16n^2}\times n\left(6\theta-16\theta^2\right)$ ***M1A1***

$=\dfrac{6\theta-16\theta^2}{16n}\ \left(=\dfrac{3\theta}{8n}-\dfrac{\theta^2}{n}\right)$

$\text{Var}(T_2)=\dfrac{1}{100n^2}\times n\left(34\theta-100\theta^2\right)$ ***M1A1***

$=\dfrac{34\theta-100\theta^2}{100n}\ \left(=\dfrac{17\theta}{50n}-\dfrac{\theta^2}{n}\right)$

attempting to form an expression for $\text{Var}(T_1)-\text{Var}(T_2)$ for example ***(M1)***

$=\dfrac{1}{16n^2}\times n\left(6\theta-16\theta^2\right)-\dfrac{1}{100n^2}\times n\left(34\theta-100\theta^2\right)$

T_2

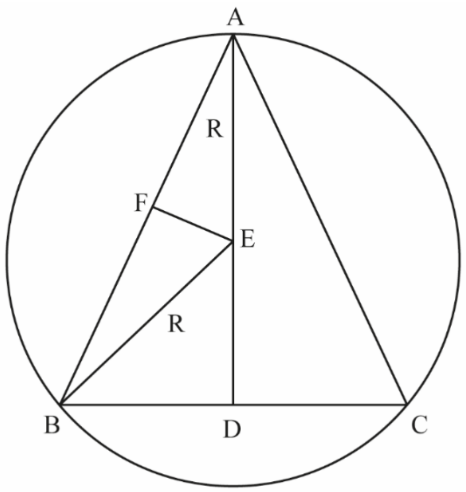
$=\dfrac{\theta}{n}\left(\dfrac{6}{16}-\theta-\dfrac{34}{100}+\theta\right)\ \left(=\dfrac{7\theta}{200n}\right)$ ***A1***

>0 therefore (T_2) is the more efficient estimator ***R1***

[7 marks]

Total [22 marks]

8. (a)



A1

METHOD 1

using the cosine rule in ΔAEB *M1*

EITHER

$$\cos\theta = \frac{R^2 + d^2 - R^2}{2Rd}$$ *A1*

$$\cos\theta = \frac{d}{2R}$$

OR

$$d^2 = 2R^2 - 2R^2\cos(180 - 2\theta)$$ *A1*

$$\Rightarrow d^2 = 2R^2(1 + \cos 2\theta)$$

THEN

$$\Rightarrow R = \frac{d}{2}\sec\theta$$ *A1*

METHOD 2

recognition that $[\text{EF}]$ is perpendicular to $[\text{AB}]$ *R1*

$$\cos\theta = \frac{\frac{d}{2}}{R}$$ *A1*

$$\Rightarrow R = \frac{d}{2}\sec\theta$$ *A1*

METHOD 3

$$R = \frac{\text{BD}}{\sin \text{B}\hat{\text{E}}\text{D}} = \frac{\text{BD}}{\sin 2\theta}$$ *M1*

$$= \frac{d\sin\theta}{2\sin\theta\cos\theta}$$ *A1*

$$= \frac{d}{2\cos\theta} \quad \left(\text{or } \frac{d}{2}\sec\theta\right)$$ *A1*

continued…

Question 8 continued

METHOD 4

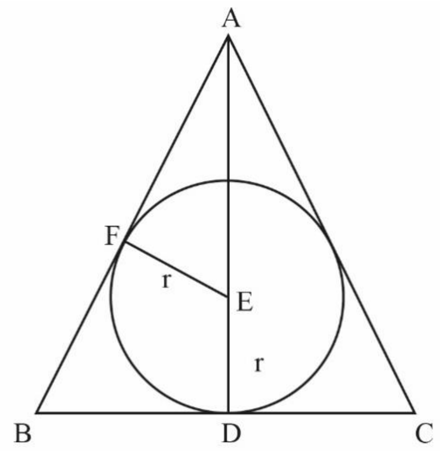
$\frac{1}{2}d^2\sin 2\theta = \frac{1}{2R}d^3\sin\theta$ ***M1***

$2R = \frac{d}{\cos\theta}$ ***A1***

$R = \frac{d}{2\cos\theta}$ (or $\frac{d}{2}\sec\theta$) ***A1***

[4 marks]

(b)



A1

METHOD 1

$\mathrm{FE} = r = \mathrm{AE}\sin\theta$ ***M1***

$\mathrm{AE} = d\cos\theta - r$ ***A1***

$r = (d\cos\theta - r)\sin\theta$

$r = d\sin\theta\cos\theta - r\sin\theta$ ***A1***

whence $r = \frac{d\sin\theta\cos\theta}{1+\sin\theta}$ ***AG***

[4 marks]

METHOD 2

$\mathrm{BD} = d\sin\theta$ and $\mathrm{AD} = r + \frac{r}{\sin\theta}$ ***(A1)***

$\mathrm{AB}^2 = \mathrm{BD}^2 + \mathrm{AD}^2$

$d^2 = d^2\sin^2\theta + \left(r + \frac{r}{\sin\theta}\right)^2$ ***M1***

$r^2 = \frac{d^2\sin^2\theta\left(1-\sin^2\theta\right)}{\left(1+\sin\theta\right)^2}$ ***A1***

whence $r = \frac{d\sin\theta\cos\theta}{1+\sin\theta}$ ***AG***

[4 marks]

continued…

Question 8 continued

METHOD 3

$\frac{1}{2}dr+\frac{1}{2}dr+\frac{1}{2}(2d\sin\theta)r=\frac{1}{2}d^2\sin 2\theta$ ***M1***

$dr+dr\sin\theta=\frac{1}{2}d^2(2\sin\theta\cos\theta)$ ***(A1)***

$r(1+\sin\theta)=d\sin\theta\cos\theta$ ***A1***

whence $r=\frac{d\sin\theta\cos\theta}{1+\sin\theta}$ ***AG***

[4 marks]

METHOD 4

$r=d\sin\theta\tan\left(45°-\frac{\theta}{2}\right)$ ***A1***

$=\frac{\left(1-\tan\frac{\theta}{2}\right)d\sin\theta}{1+\tan\frac{\theta}{2}}$

$=\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\sin\theta}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}$ ***M1***

Note: Award ***M1*** for attempting to express with terms involving $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$.

$=\frac{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\sin\theta}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}\left(=\frac{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\sin\theta}{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$ ***A1***

whence $r=\frac{d\sin\theta\cos\theta}{1+\sin\theta}$ ***AG***

[4 marks]

(c) $\mathrm{FE}=\rho=\mathrm{AE}\sin\theta$ ***(M1)***

$\mathrm{AE}=d\cos\theta+\rho$ ***(A1)***

$\rho=(d\cos\theta+\rho)\sin\theta$

$\rho=d\sin\theta\cos\theta+\rho\sin\theta$ ***(A1)***

whence $\rho=\frac{d\sin\theta\cos\theta}{1-\sin\theta}$ ***A1***

[4 marks]

continued…

Question 8 continued

(d) **METHOD 1**

if ΔABC is equilateral then $\theta = 30°$

$r = \frac{d}{2\sqrt{3}}$ ***A1***

$R = \frac{d}{\sqrt{3}}$ ***A1***

$\rho = \frac{3d}{2\sqrt{3}}$ ***A1***

these radii form an arithmetic sequence because $R - r = \rho - R = \frac{d}{2\sqrt{3}}$ ***R1***

[4 marks]

METHOD 2

using $2R = r + \rho$ (or equivalent) ***(M1)***

$2R = \frac{d}{\cos\theta}$ and $r + \rho = \frac{d\sin\theta\cos\theta}{1+\sin\theta} + \frac{d\sin\theta\cos\theta}{1-\sin\theta}$

$r + \rho = 2d\tan\theta$ ***A1***

EITHER

solving $\frac{d}{\cos\theta} = 2d\tan\theta$ for θ gives $\theta = 30°$ (so ΔABC is equilateral) ***A1***

OR

if ΔABC is equilateral then $\theta = 30°$

$\frac{d}{\cos 30°} = \frac{2d}{\sqrt{3}}$ and $2d\tan 30° = \frac{2d}{\sqrt{3}}$ ***A1***

THEN

these radii form an arithmetic sequence because $2R = r + \rho$ (or equivalent) ***R1***

[4 marks]

Total [16 marks]

9. (a) (i) the auxiliary equation is $m^2 - 4m + 5 = 0$ (or equivalent) ***A1***
attempting to solve their auxiliary equation ***(M1)***
$m = 2 \pm \mathrm{i}$ ***A1***

(ii) the general solution is
$u_n = A(2+\mathrm{i})^n + B(2-\mathrm{i})^n$ ***A1***

(iii) applying both initial conditions, ***M1***
$A + B = 2$; $A(2+\mathrm{i}) + B(2-\mathrm{i}) = 2$ ***A1***
attempting to solve their two equations ***M1***
$A = 1 + \mathrm{i}$, $B = 1 - \mathrm{i}$ ***A1***

Note: Award the ***M1*** for a convincing attempt to solve their two equations.

therefore
$u_n = (1+\mathrm{i})(2+\mathrm{i})^n + (1-\mathrm{i})(2-\mathrm{i})^n$ ***AG***

[8 marks]

(b) attempting to convert to polar form ***(M1)***

$$1 \pm \mathrm{i} = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \mathrm{i}\sin\frac{\pi}{4}\right),$$ ***(A1)***

$$2 \pm \mathrm{i} = \sqrt{5}\left(\cos\arctan\frac{1}{2} \pm \mathrm{i}\sin\arctan\frac{1}{2}\right)$$ ***(A1)***

$$u_n = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)\left(\sqrt{5}\right)^n\left(\cos n\arctan\frac{1}{2} + \mathrm{i}\sin n\arctan\frac{1}{2}\right)$$

$$+\sqrt{2}\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right)\left(\sqrt{5}\right)^n\left(\cos n\arctan\frac{1}{2} - \mathrm{i}\sin n\arctan\frac{1}{2}\right)$$ ***M1A1***

$$= \sqrt{2}\left(\sqrt{5}\right)^n\left(\cos\left(\frac{\pi}{4} + n\arctan\frac{1}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + n\arctan\frac{1}{2}\right)\right)$$

$$+\sqrt{2}\left(\sqrt{5}\right)^n\left(\cos\left(\frac{\pi}{4} + n\arctan\frac{1}{2}\right) - \mathrm{i}\sin\left(\frac{\pi}{4} + n\arctan\frac{1}{2}\right)\right)$$ ***M1A1***

$$= 2^{\frac{3}{2}}5^{\frac{n}{2}}\cos\left(\frac{\pi}{4} + n\arctan\left(\frac{1}{2}\right)\right)$$ ***AG***

[7 marks]

Total [15 marks]