Oxford Cambridge and RSA

Wednesday 5 June 2019 – Morning

A Level Mathematics A

H240/01 Pure Mathematics

Time allowed: 2 hours

* 7 6 6 2 3 3 7 6 5 2 *

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + c$$

Integration by parts $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$ **or** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

Solve the inequality $10x^2 + x - 2 > 0$. **[4]**

2 The point A is such that the magnitude of $\overrightarrow{OA}$ is 8 and the direction of $\overrightarrow{OA}$ is 240°.

(a) **(i)** Show the point A on the axes provided in the Printed Answer Booklet. **[1]**

(ii) Find the position vector of point A.
Give your answer in terms of **i** and **j**. **[3]**

The point B has position vector 6**i**.

(b) Find the exact area of triangle AOB. **[2]**

The point C is such that $OABC$ is a parallelogram.

(c) Find the position vector of C.
Give your answer in terms of **i** and **j**. **[2]**

3 The function f is defined by $\mathrm{f}(x) = (x-3)^2 - 17$ for $x \geqslant k$, where k is a constant.

(a) Given that $\mathrm{f}^{-1}(x)$ exists, state the least possible value of k. **[1]**

(b) Evaluate ff(5). **[2]**

(c) Solve the equation $\mathrm{f}(x) = x$. **[3]**

(d) Explain why your solution to part **(c)** is also the solution to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[1]**

4 Sam starts a job with an annual salary of £16 000. It is promised that the salary will go up by the same amount every year. In the second year Sam is paid £17 200.

(a) Find Sam's salary in the tenth year. **[2]**

(b) Find the number of complete years needed for Sam's **total** salary to first exceed £500 000. **[4]**

(c) Comment on how realistic this model may be in the long term. **[1]**

5 A curve has equation $x^3 - 3x^2y + y^2 + 1 = 0$.

(a) Show that $\frac{dy}{dx} = \frac{6xy - 3x^2}{2y - 3x^2}$. **[4]**

(b) Find the equation of the normal to the curve at the point (1, 2). **[4]**

6 Let $f(x) = 2x^3 + 3x$. Use differentiation from first principles to show that $f'(x) = 6x^2 + 3$. **[6]**

7 **In this question you must show detailed reasoning.**

A sequence $u_1, u_2, u_3 \ldots$ is defined by $u_n = 25 \times 0.6^n$.

Use an algebraic method to find the smallest value of N such that $\sum_{n=1}^{\infty} u_n - \sum_{n=1}^{N} u_n < 10^{-4}$. **[8]**

8 A cylindrical tank is initially full of water. There is a small hole at the base of the tank out of which the water leaks.

The height of water in the tank is x m at time t seconds. The rate of change of the height of water may be modelled by the assumption that it is proportional to the square root of the height of water.

When $t = 100$, $x = 0.64$ and, at this instant, the height is decreasing at a rate of $0.0032\,\text{m s}^{-1}$.

(a) Show that $\frac{dx}{dt} = -0.004\sqrt{x}$. **[2]**

(b) Find an expression for x in terms of t. **[4]**

(c) Hence determine at what time, according to this model, the tank will be empty. **[2]**

9 **(a)** Express $3\cos 3x + 7\sin 3x$ in the form $R\cos(3x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. **[3]**

(b) Give full details of a sequence of three transformations needed to transform the curve $y = \cos x$ to the curve $y = 3\cos 3x + 7\sin 3x$. **[4]**

(c) Determine the **greatest** value of $3\cos 3x + 7\sin 3x$ as x varies and give the smallest positive value of x for which it occurs. **[2]**

(d) Determine the **least** value of $3\cos 3x + 7\sin 3x$ as x varies and give the smallest positive value of x for which it occurs. **[2]**

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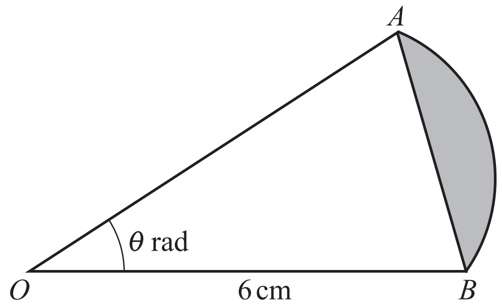

The diagram shows a sector AOB of a circle with centre O and radius 6 cm.
The angle AOB is θ radians.
The area of the segment bounded by the chord AB and the arc AB is $7.2\,\text{cm}^2$.

(a) Show that $\theta = 0.4 + \sin\theta$. **[3]**

(b) Let $F(\theta) = 0.4 + \sin\theta$.

By considering the value of $F'(\theta)$ where $\theta = 1.2$, explain why using an iterative method based on the equation in part **(a)** will converge to the root, assuming that 1.2 is sufficiently close to the root. **[2]**

(c) Use the iterative formula $\theta_{n+1} = 0.4 + \sin\theta_n$ with a starting value of 1.2 to find the value of θ correct to 4 significant figures.
You should show the result of each iteration. **[3]**

(d) Use a change of sign method to show that the value of θ found in part **(c)** is correct to 4 significant figures. **[3]**

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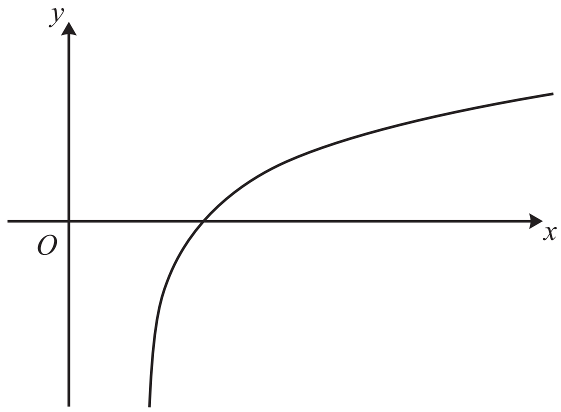

The diagram shows part of the curve $y = \ln(x-4)$.

(a) Use integration by parts to show that $\int \ln(x-4)\,dx = (x-4)\ln|x-4| - x + c$. **[5]**

(b) State the equation of the vertical asymptote to the curve $y = \ln(x-4)$. **[1]**

(c) Find the total area enclosed by the curve $y = \ln(x-4)$, the x-axis and the lines $x = 4.5$ and $x = 7$. Give your answer in the form $a\ln 3 + b\ln 2 + c$ where a, b and c are constants to be found. **[4]**

12 A curve has equation $y = a^{3x^2}$, where a is a constant greater than 1.

(a) Show that $\frac{dy}{dx} = 6xa^{3x^2}\ln a$. **[3]**

(b) The tangent at the point $(1, a^3)$ passes through the point $(\frac{1}{2}, 0)$.

Find the value of a, giving your answer in an exact form. **[4]**

(c) By considering $\frac{d^2y}{dx^2}$ show that the curve is convex for all values of x. **[5]**

END OF QUESTION PAPER