OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT

4777/01

MATHEMATICS (MEI)

Numerical Computation

FRIDAY 22 JUNE 2007

Morning
Time: 2 hours 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the answer booklet.

COMPUTER RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities during the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the *formulae* in the cells as well as the *values* in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

1 **(i)** The iterative sequence $x_0, x_1, x_2, \ldots$, where $x_{r+1} = g(x_r)$, has a fixed point α, so that $\alpha = g(\alpha)$.

Given that $x_{r+1} - \alpha \approx k(x_r - \alpha)$, show that

$$k \approx \frac{(x_2 - x_1)}{(x_1 - x_0)},$$

and obtain an approximate equation for α in terms of x_2, x_1 and k. [5]

(ii) Use a spreadsheet to demonstrate graphically that the equation $x = f(x)$ where

$$f(x) = e^{\frac{1}{3}x} + e^{-\frac{1}{3}x} - 0.5$$

has two roots. Let these roots be α and β where $\alpha < \beta$.

Show that the iteration $x_{r+1} = f(x_r)$ will converge only slowly to α and that it will not converge to β at all. [7]

(iii) Use the acceleration technique developed in part **(i)** to speed up the convergence to α. Find α correct to 6 decimal places.

Show that, with a carefully chosen starting point, the acceleration technique may be used to produce convergence to β. Find β correct to 6 decimal places.

Determine, correct to 1 decimal place, the range of starting values for which convergence to β is assured within 5 iterations of the acceleration technique. [12]

2 The Gaussian 4-point integration formula has the form

$$\int_{-h}^{h} f(x)\,dx = af(-\alpha) + bf(-\beta) + bf(\beta) + af(\alpha).$$

(i) Obtain the four equations that determine a, b, α and β, showing that one of them is

$$a\alpha^6 + b\beta^6 = \tfrac{1}{7}h^7.$$ [7]

You are now *given* the following values, correct to 8 decimal places.

$$a = 0.347\,854\,85h$$
$$b = 0.652\,145\,15h$$
$$\alpha = 0.861\,136\,31h$$
$$\beta = 0.339\,981\,04h$$

(ii) Use a spreadsheet to show that, for x in radians, $\dfrac{\sin x}{x}$ tends to 1 as x tends to 0.

Use a spreadsheet to obtain a sketch of the function $f(x) = \dfrac{\sin x}{x}$ for $0 \leqslant x \leqslant \pi$.

Taking $h = \frac{1}{2}\pi$ initially, use the Gaussian 4-point rule to estimate the value of

$$\int_0^{\pi} \frac{\sin x}{x}\,dx.$$

Repeat the process, halving h as necessary, in order to establish the value of the integral correct to 6 decimal places. [13]

(iii) Modify the routines used in part **(ii)** to determine the value of t, correct to 3 decimal places, such that

$$\int_0^{t} \frac{\sin x}{x}\,dx = 1.$$ [4]

3 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + 0.1\mathrm{e}^{y},$$

where $y = 0$ when $x = 0$, is to be solved in order to estimate y when $x = 1$.

(i) Use Euler's method with $h = 0.2, 0.1, 0.05, 0.025$ to obtain a sequence of estimates of y when $x = 1$. Hence demonstrate that Euler's method has first order convergence. [7]

(ii) Show similarly that the modified Euler method has second order convergence. [6]

(iii) Develop a solution to the differential equation using a predictor-corrector method. Use Euler's method as the predictor and the modified Euler method as the corrector. Apply the corrector 3 times at each step.

Compare the accuracy of this method with that of the modified Euler method. [8]

(iv) Obtain a sequence of estimates of y when $x = 1$ by averaging the estimates found in parts **(ii)** and **(iii)**. Show that this sequence appears to have approximately third order convergence. [3]

4 The augmented matrix given below is denoted by $\mathbf{M} \mid \mathbf{c}$.

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 1 \\ 3 & 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 2 & 3 & 0 & 4 \end{array}\right)$$

(i) Set up a spreadsheet using Gaussian elimination to solve the system of equations represented by $\mathbf{M} \mid \mathbf{c}$. Make clear at each stage which element is used for pivoting and explain why. Show how to check the accuracy of your solution. [13]

(ii) Apply the routine developed in part **(i)** to systems of the form $\mathbf{M} \mid \mathbf{v}$, for appropriate vectors $\mathbf{v}$ so as to find the inverse of the matrix $\mathbf{M}$. [6]

(iii) Use part **(i)** to obtain the determinant of $\mathbf{M}$, making it clear how you establish its sign. [5]