OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4777

Numerical Computation

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)
- Graph paper

Other Materials Required:
None

Tuesday 23 June 2009
Morning

Duration: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.
  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.
  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 **(i)** The equation $x = \mathrm{g}(x)$ has a root $x = \alpha$. State a condition on the derivative of $\mathrm{g}(x)$ that will ensure convergence of the iteration $x_{r+1} = \mathrm{g}(x_r)$ provided x_0 is close enough to α.

Obtain the relaxed iteration $x_{r+1} = \lambda\,\mathrm{g}(x_r) + (1 - \lambda)\,x_r$. Show that, for fastest convergence,

$$\lambda = \frac{1}{1 - \mathrm{g}'(\alpha)}.$$

State how a value for λ would be chosen in practice. **[7]**

(ii) Use a spreadsheet to show graphically that the equation

$$x = 3\sin x - 0.5$$

(where x is in radians) has two roots in the interval (0, 3). Use your graph to give approximate values for these roots.

Show that the iteration

$$x_{r+1} = 3\sin x_r - 0.5$$

does not converge to either root. You should try several values of x_0 in each case.

Use the method of relaxation to find each root correct to 6 decimal places. **[17]**

2 The Gaussian 3-point integration formula has the form

$$\int_{-h}^{h} \mathrm{f}(x)\,\mathrm{d}x = a\,\mathrm{f}(-\alpha) + b\,\mathrm{f}(0) + a\,\mathrm{f}(\alpha).$$

(i) By considering $\mathrm{f}(x) = 1, x, x^2, x^3, x^4$, obtain the three equations that determine a, b and α. Verify that these equations are satisfied by

$$\alpha = \sqrt{\tfrac{3}{5}}\,h,$$

$$a = \tfrac{5}{9}\,h,$$

$$b = \tfrac{8}{9}\,h.$$ **[8]**

(ii) Taking $h = \frac{\pi}{8}$ initially, use the Gaussian 3-point rule to estimate the value of

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan x}\,\mathrm{d}x.$$

Repeat the process, halving h as necessary, in order to establish the value of the integral correct to 6 decimal places. **[12]**

(iii) Determine, correct to 3 decimal places, the value of k such that

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + k\tan x}\,\mathrm{d}x = 1.$$ **[4]**

3 The second order differential equation

$$\frac{d^2y}{dx^2} + \sqrt{x}\,\frac{dy}{dx} + xy = 1$$

with initial conditions $x = 0$, $y = 0$, $\frac{dy}{dx} = a$, is to be solved for various values of a using finite difference methods.

(i) Consider first the case $a = 1$.

Show that, in the usual notation,

$$y_{r+1} = \frac{2(2 - h^2 x_r)\,y_r + (h\sqrt{x_r} - 2)\,y_{r-1} + 2h^2}{2 + h\sqrt{x_r}},$$

and that

$$y_1 = h + \tfrac{1}{2}h^2. \qquad (*)$$ **[8]**

(ii) Obtain a solution from $x = 0$ to $x = 5$ with $h = 0.1$. Use your spreadsheet to produce a graph of this solution. **[9]**

(iii) Modify (*) to allow different values of a to be used.

Still using $h = 0.1$, find, correct to 1 decimal place, a negative value of a for which the graph of the solution curve crosses the axis very close to $x = 2$. **[7]**

4 The system of linear equations with augmented matrix

$$\left(\begin{array}{cccc|c} a & 1 & b & 1 & 1 \\ 1 & a & 1 & b & 0 \\ b & 1 & a & 1 & 0 \\ 1 & b & 1 & a & 0 \end{array}\right)$$

is to be solved, using the Gauss-Seidel method, for various values of a and b.

(i) Explain the condition of diagonal dominance. State a condition on a and b that will ensure convergence. **[3]**

(ii) Set up a spreadsheet implementing the Gauss-Seidel method and allowing the user to vary the values of a and b.

Show that convergence does occur in the case $a = 4$, $b = 2$, and does not occur in the case $a = 2$, $b = 4$. **[12]**

(iii) Investigate the case $a = 2$, $b = 0$. What do your results indicate about diagonal dominance? **[4]**

(iv) By modifying your spreadsheet find the inverse of the following matrix.

$$\begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

[5]