

Mathematics (MEI)

Advanced GCE

Unit 4756: Further Methods for Advanced Mathematics

Mark Scheme for January 2012

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

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Annotations

Annotation	Meaning
✓ and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	Meaning
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M

A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A

Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B

Mark for a correct result or statement independent of Method marks.

E

A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

- d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.
- e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.
- f Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.
- g Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.
- h Rules for replaced work

If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

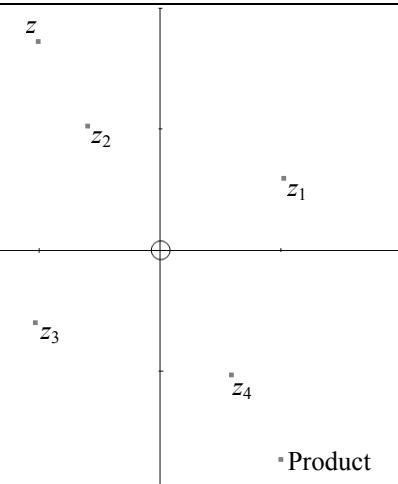
- h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

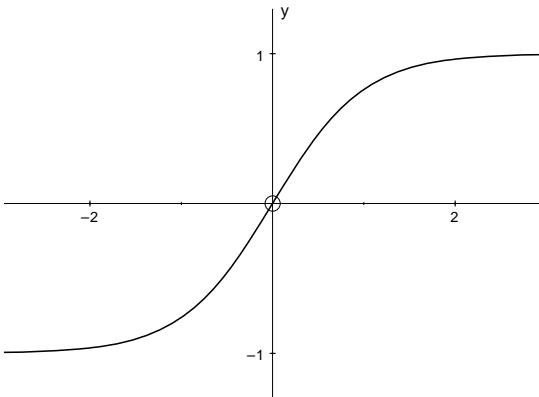
Question			Answer	Marks	Guidance	
1	(a)	(i)		G2 [2]	A fully correct curve Give G1 for one error, e.g. incorrect form at O, lack of clear symmetry, sharp point at RH extremity	
1	(a)	(ii)	$\text{Area} = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta$ $= \frac{1}{2} \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$ $= \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\cos \theta + \frac{1}{2}\cos 2\theta \right) d\theta$ $= \frac{1}{2} \left[\frac{3}{2}\theta + 2\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi}$ $= \frac{3}{2}\pi$	M1 A1 M1 A2 A1 [6]	Integral expression involving $(1 + \cos \theta)^2$ Correct expanded integral expression, incl. limits Using $\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ Correct result of integration Dependent on previous A2	Limits may be implied by later work. Penalise missing $\frac{1}{2}$ here (max. 4/6) Allow sign or factor errors Give A1 for one error in this expression
1	(b)		$\sin x = x - \frac{1}{6}x^3 \dots$ $\cos x = 1 - \frac{1}{2}x^2 \dots$ $\tan x \approx \left(x - \frac{1}{6}x^3\right) \left(1 - \frac{1}{2}x^2\right)^{-1}$	B1 M1	Both series correct as far as second term Using $\tan x = \frac{\sin x}{\cos x}$	Ignore higher-order terms. Allow denominators left as 2!, 3! Allow even if no further progress but must be used, not just stated

Question		Answer	Marks	Guidance		
		$= \left(x - \frac{1}{6}x^3 \right) \left(1 + \frac{1}{2}x^2 + \dots \right)$ $= x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \dots$ $= x + \frac{1}{3}x^3 \dots$	M1 M1 A1A1	Using binomial expansion. Dependent on first M1 Expanding brackets. Dependent on previous M1 $a = 1, b = \frac{1}{3}$ correctly obtained	If methods mixed, mark to benefit of candidate Dependent on both M1s. Deduct 1 for each additional term (*)	
		OR $\frac{x - \frac{1}{6}x^3}{1 - \frac{1}{2}x^2} = ax + bx^3$ $\Rightarrow x - \frac{1}{6}x^3 = \left(1 - \frac{1}{2}x^2 \right) (ax + bx^3) = ax + \left(b - \frac{1}{2}a \right)x^3 + \dots$ $\Rightarrow a = 1$ $b - \frac{1}{2}a = -\frac{1}{6}$ $\Rightarrow b = \frac{1}{3}$	M1 M1 A1 M1 A1	Using $\tan x = \frac{\sin x}{\cos x}$ Attempting to compare coeffs. Correctly obtained Obtaining b Correctly obtained	As (*) As (*)	
		OR $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ $f''(x) = 2 \sec^2 x \tan x$ $f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 2$ $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \dots$ $= x + \frac{1}{3}x^3 \dots$	M1 M1 M1 A1A1	Attempting first two derivatives Attempting third derivative Applying Maclaurin series. Dependent on first M1 Correctly obtained	Using the product rule As (*)	
1 (c)		$\int_0^1 \frac{1}{\sqrt{1 - \frac{1}{4}x^2}} dx = \int_0^1 \frac{2}{\sqrt{4 - x^2}} dx = \left[2 \arcsin \frac{x}{2} \right]_0^1$ $= 2 \left(\frac{\pi}{6} - 0 \right)$ $= \frac{\pi}{3}$	[6]	M1 A1 M1 A1 [4]	arcsin alone, or any sine substitution 2 and $\frac{x}{2}$ Using limits. Dependent on first M1 Evaluated in terms of π	No need to see explicit use of $x = 0$ Limits wrong way round M0

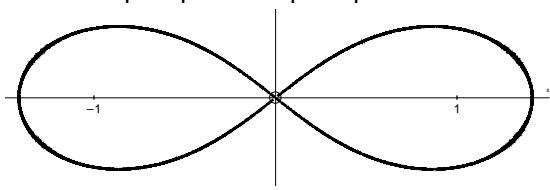
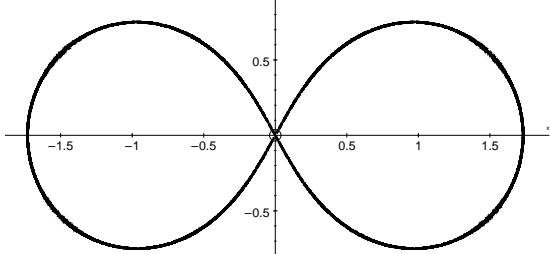
Question		Answer	Marks	Guidance
2	(a)	$C + jS = 1 + ae^{j\theta} + a^2 e^{2j\theta} + \dots$ <p>This is a geometric series with $r = ae^{j\theta}$</p> $\text{Sum to infinity} = \frac{1}{1 - ae^{j\theta}}$ $= \frac{1}{1 - ae^{j\theta}} \times \frac{1 - ae^{-j\theta}}{1 - ae^{-j\theta}}$ $= \frac{1 - ae^{-j\theta}}{1 - ae^{j\theta} - ae^{-j\theta} + a^2}$ $= \frac{1 - a(\cos \theta - j \sin \theta)}{1 - 2a \cos \theta + a^2}$ $= \frac{1 - a \cos \theta}{1 - 2a \cos \theta + a^2} + \frac{aj \sin \theta}{1 - 2a \cos \theta + a^2}$ $\Rightarrow C = \frac{1 - a \cos \theta}{1 - 2a \cos \theta + a^2}$ <p>and $S = \frac{a \sin \theta}{1 - 2a \cos \theta + a^2}$</p>	M1 M1 A1 M1* M1 M1 E1 A1 [8]	Forming $C + jS$ as a series of powers Identifying G.P. and attempting sum. Dependent on first M1 Multiplying numerator and denominator by $1 - ae^{-j\theta}$ o.e. Multiplying out denominator. Dependent on M1* Introducing trig functions. Dependent on M1* Answer given. www which leads to C
2	(b)	Modulus = 2 Argument = $\frac{2\pi}{3}$ $\Rightarrow -1 + j\sqrt{3} = 2e^{j\frac{2\pi}{3}}$ \Rightarrow fourth roots have $r = \sqrt[4]{2}$ and $\theta = \frac{\pi}{6}$ \Rightarrow roots are $\sqrt[4]{2}e^{j\frac{\pi}{6}}, \sqrt[4]{2}e^{j\frac{2\pi}{3}}, \sqrt[4]{2}e^{j\frac{7\pi}{6}}, \sqrt[4]{2}e^{j\frac{5\pi}{3}}$	B1 B1 B1 M1 A1	Allow 1.19 or better $\div \arg z \text{ by 4 and adding } \frac{\pi}{2}$ $\theta = \frac{\pi}{6} + \frac{2k\pi}{4}$ scores M1; $k = 0, 1, 2, 3$ (or $-2, -1, 0, 1$) A1

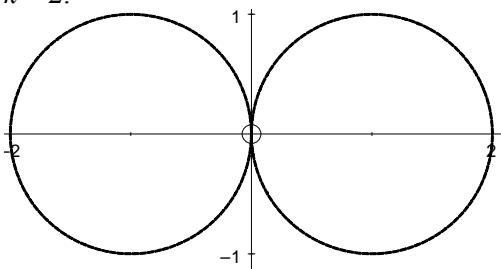
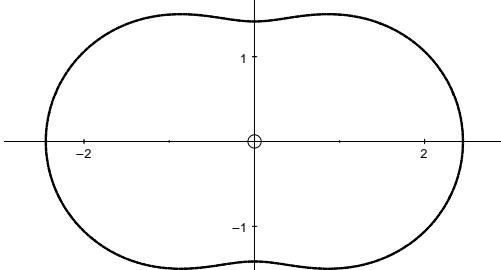
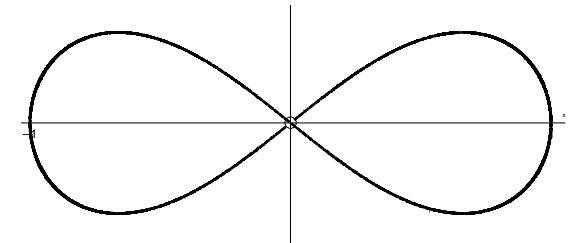
Question		Answer	Marks	Guidance
		 <p>Product</p> <p>Product of 4th roots = $2e^{j(1+4+7+10)\frac{\pi}{6}}$</p> <p>= $2e^{j\frac{5\pi}{3}}$</p>	G1 G1ft G1ft M1 A1 [10]	Position of z Roots forming square Position of product Attempting to find product Or $-\frac{\pi}{3}$ o.e.
3	(i)	$\mathbf{M} - \lambda \mathbf{I} = \begin{pmatrix} 3-\lambda & -1 & 2 \\ -4 & 3-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{pmatrix}$ $\det(\mathbf{M} - \lambda \mathbf{I}) = (3-\lambda)[(3-\lambda)(-1-\lambda) - 2]$ $+ 1[-4(-1-\lambda) - 4] + 2[-4 - 2(3-\lambda)]$ $= (3-\lambda)(\lambda^2 - 2\lambda - 5) + 4\lambda + 2(2\lambda - 10)$ $= -\lambda^3 + 5\lambda^2 - \lambda - 15 + 4\lambda + 4\lambda - 20$ $\Rightarrow \lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$	M1 A1 M1 E1 [4]	Obtaining $\det(\mathbf{M} - \lambda \mathbf{I})$ Any correct form Multiplying out. Dep. on first M1 Answer given
3	(ii)	$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$ $\Rightarrow (\lambda - 5)(\lambda^2 - 7) = 0$ $\lambda = \pm\sqrt{7}$	M1 A1 M1 A1 [4]	Factorising, obtaining a quadratic Correct quadratic Solving quadratic If M0, give B1 for substituting $\lambda = 5$ Allow 2.65 or better

Question		Answer	Marks	Guidance	
3	(iii)	$\lambda = 5 \Rightarrow \begin{pmatrix} -2 & -1 & 2 \\ -4 & -2 & 2 \\ 2 & 1 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow -2x - y + 2z = 0$ $-4x - 2y + 2z = 0$ $2x + y - 6z = 0$ $\Rightarrow z = 0, y = -2x$ $\Rightarrow \text{eigenvector is } \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ $\Rightarrow \text{eigenvector of unit length is } \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ $\mathbf{M}^2 \mathbf{v} \text{ has magnitude 25 in direction of } \mathbf{v}$ $\mathbf{M}^{-1} \mathbf{v} \text{ has magnitude } \frac{1}{5} \text{ in direction of } \mathbf{v}$	M1 M1 A1 A1ft B1 B1 [6]	Two independent equations Obtaining a non-zero eigenvector $\frac{1}{\sqrt{5}}$ f.t. their eigenvector Both magnitudes c.a.o. Directions c.a.o.	Need to multiply out, unless implied by later work May be given as column vectors
3	(iv)	$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$ $\Rightarrow \mathbf{M}^3 - 5\mathbf{M}^2 - 7\mathbf{M} + 35\mathbf{I} = \mathbf{0}$ $\Rightarrow \mathbf{M}^4 = 5\mathbf{M}^3 + 7\mathbf{M}^2 - 35\mathbf{M}$ $= 5(5\mathbf{M}^2 + 7\mathbf{M} - 35\mathbf{I}) + 7\mathbf{M}^2 - 35\mathbf{M}$ $= 32\mathbf{M}^2 - 175\mathbf{I}$	M1 A1 M1 A1 [4]	Using Cayley-Hamilton Theorem Correct expression involving \mathbf{M}^4 and non-negative powers of \mathbf{M} Substituting for \mathbf{M}^3 and obtaining expression in required form $a = 32, b = 0, c = -175$	Condone omitted \mathbf{I}

Question		Answer	Marks	Guidance	
4	(i)	$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ 	B1 G1 G1 [3]	Or $\frac{e^{2t} - 1}{e^{2t} + 1}$ Correct shape Asymptotes at $y = \pm 1$. Dependent on first G1	Condone other variables used If text and graph conflict, mark what is shown on the graph
4	(ii)	$y = \operatorname{artanh} x \Rightarrow x = \tanh y$ $\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$ $\Rightarrow x(e^y + e^{-y}) = e^y - e^{-y}$ $\Rightarrow xe^y + xe^{-y} = e^y - e^{-y}$ $\Rightarrow xe^{-y} + e^{-y} = e^y - xe^y$ $\Rightarrow e^{-y}(1+x) = e^y(1-x)$ $\Rightarrow e^{2y} = \frac{1+x}{1-x}$ $\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$ $\Rightarrow \operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ <p>Valid for $-1 < x < 1$</p>	M1 M1 A1 E1 B1 [5]	First step in rearrangement Obtaining e^{2y} in terms of x	Or $x = \frac{e^{2y} - 1}{e^{2y} + 1}$ Variables the right way round at some stage, and clearing fractions Dependent on first M1 Answer given

Question		Answer	Marks	Guidance	
4	(iii)	$\tanh y = x$ $\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$ $= \frac{1}{1 - x^2}$ $y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x)$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{1+x} - \frac{1}{2} \times \frac{-1}{1-x}$ $= \frac{1}{2} \times \frac{1-x+1+x}{(1+x)(1-x)}$ $= \frac{1}{1-x^2}$	M1 E1 M1 A1 E1 [5]	Differentiating and explicitly attempting to express in terms of $\tanh y$ Correctly obtained Attempting logarithmic diff. Any correct form Convincing manipulation	
4	(iv)	$\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx = \left[x \operatorname{artanh} x \right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2} \, dx$ $= \frac{1}{2} \operatorname{artanh} \frac{1}{2} - \left[-\frac{1}{2} \ln(1-x^2) \right]_0^{\frac{1}{2}}$ $= \frac{1}{4} \ln \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) + \frac{1}{2} \ln \frac{3}{4}$ $= \frac{1}{4} \ln 3 + \frac{1}{4} \ln \frac{9}{16}$ $= \frac{1}{4} \ln \frac{27}{16}$	M1 A1 A1 M1 E1 [5]	Using integration by parts This line correct $-\frac{1}{2} \ln(1-x^2)$ Applying limits and using (ii) in result of integration Convincing manipulation www	With $u = \operatorname{artanh} x$, $v' = 1$ and $v = x$ Condone omitted limits Or $-\frac{1}{2} \ln(1-x) - \frac{1}{2} \ln(1+x)$ Must be exact Answer given

Question		Answer	Marks	Guidance	
5	(i)	$((x+1)^2 + y^2)((x-1)^2 + y^2) = 1$ $\Rightarrow (x^2 + 2x + 1 + y^2)(x^2 - 2x + 1 + y^2) = 1$ $\Rightarrow (r^2 + 2r \cos \theta + 1)(r^2 - 2r \cos \theta + 1) = 1$ $\Rightarrow (r^2 + 1)^2 - 4r^2 \cos^2 \theta = 1$ $\Rightarrow r^4 + 2r^2 = 4r^2 \cos^2 \theta$ $\Rightarrow r^4 = 2r^2(2 \cos^2 \theta - 1)$ $\Rightarrow r^4 = 2r^2 \cos 2\theta$ $\Rightarrow r^2 = 2 \cos 2\theta$	M1 E1 M1 E1 [4]	Using $r^2 = x^2 + y^2$ and $x = r \cos \theta$ Correctly obtained Using $\cos 2\theta = 2 \cos^2 \theta - 1$ Correctly obtained	
5	(ii)	$r^2 \geq 0 \Rightarrow \cos 2\theta \geq 0$ $\Rightarrow 0 \leq \theta \leq \frac{\pi}{4}, \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}, \frac{7\pi}{4} \leq \theta \leq 2\pi$ 	M1 A1 G2 [4]	Considering $\cos 2\theta \geq 0$ Or B2. Inequalities must be non-strict Curve must be complete. Award G1 for a curve with one error	
5	(iii)	(A)	$k = 1:$ 	G1	Curve must be complete

Question		Answer	Marks	Guidance
	(B)	<p>$k = 2$:</p>  <p>For $k = 1$, the gradients at the pole are finite For $k = 2$, they appear to be infinite</p> <p>$k = 4$:</p>  <p>Tends to circle as k tends to infinity</p>	G2 B1	Curve must be complete. Award G1 for a curve with one error For G2, curve must appear sufficiently different from case $k = 1$
5	(iv)	<p>$k = -1$:</p>  <p>As $k \rightarrow -2$, the curve retains its figure-of-eight shape, but contracts towards the origin</p>	G2 B1 [3]	Curve must be complete. Award G1 for a curve with one error

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