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MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4) Paper B: Comprehension

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FRIDAY 23 JUNE 2017: Morning
DURATION: Up to 1 hour
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Feigenbaum's Constant

A population model

A remote island has a population of squirrels. There are no predators so that the population is limited only by the supply of food and death from natural causes.

How would the number of squirrels be expected to change from year to year?

A simple model involves the following variables.

Time is measured in years and the year number is denoted by n. In this article, year n is also described as 'this year' and year $n+1$ as 'next year'.

The size of the squirrel population is given as the proportion of the maximum possible population; the population is given at the start of each year. The population in year n is denoted by x_n where $0 \leqslant x_n \leqslant 1$. Similarly, in year $n+1$ the population is x_{n+1}.

A parameter k is a measure of the reproductivity of the squirrels, and so determines the rate of growth of their population.

The model involves the following assumptions.

The number of squirrels next year, x_{n+1}, is jointly proportional to the number this year, x_n, and to the quantity $(1 - x_n)$ which represents the food available.

The parameter k is the constant of proportionality.

The model can be expressed by the iterative equation

$$x_{n+1} = kx_n(1 - x_n).$$

This is a general model for a population in a restricted environment without predators. The animals do not have to be squirrels.

Different values of k

To investigate this model, it is first necessary to choose values for k and for the initial population, x_0. Table 1 gives the first ten values of x_n for four particular values of k and with $x_0 = 0.5$. The numbers given in the table have been truncated; many more figures were used in calculating them.

TABLE 1

	$x_{n+1} = kx_n(1 - x_n)$			
	$k = 0.6$	$k = 1.5$	$k = 3.3$	$k = 4.5$
x_0	0.5	0.5	0.5	0.5
x_1	0.15	0.375	0.825	1.125
x_2	0.0765	0.3515...	0.4764...	−0.6328...
x_3	0.0423...	0.3419...	0.8231...	−4.649...
x_4	0.0243...	0.3375...	0.4803...	−118.2...
x_5	0.0142...	0.3354...	0.8237...	–
x_6	0.0084...	0.3343...	0.4791...	–
x_7	0.0050...	0.3338...	0.8235...	–
x_8	0.0029...	0.3335...	0.4795...	–
x_9	0.0017...	0.3334...	0.8236...	–
x_{10}	0.0010...	0.3333...	0.4794...	–
...	...	...	...	...

The different values of k used in Table 1 lead to four different outcomes for the population in this model.

When $k = 0.6$, $x_n \to 0$. Eventually the population dies out.

When $k = 1.5$, $x_n \to 0.3333... = \frac{1}{3}$. The population attains a stable equilibrium level.

When $k = 3.3$, the population alternates between a high value of 0.823... and a low value of 0.479... in successive years.

When $k = 4.5$, the population appears to become negative and so to have died out. In the first year the population exceeds the limit of 1 and so the model has broken down.

In each case in Table 1 the value of x_0 was taken to be 0.5, the middle of the possible values of x_0. If you try other starting values you will find that the final outcomes are the same for any values of x_0 between, but not including, 0 and 1.

Overall the model suggests that having too few or too many young can both be fatal for the population.

Possible outcomes

This iteration can be used as a population model, but it can also be thought of as a mathematical iteration in its own right with an interesting variety of possible outcomes.

At this stage it is helpful to extend the notation used to include the letter x. This denotes the value, or values, of x_n as n tends to infinity.

In Fig. 2 opposite, these limiting values, x, are plotted for values of k between 0 and 3.6. For larger values of k there are no limiting values. It is assumed that $0 < x_0 < 1$.

Fig. 2 shows that, for this iteration, five different types of outcome are possible according to the value of k. These are described below.

Tending to zero

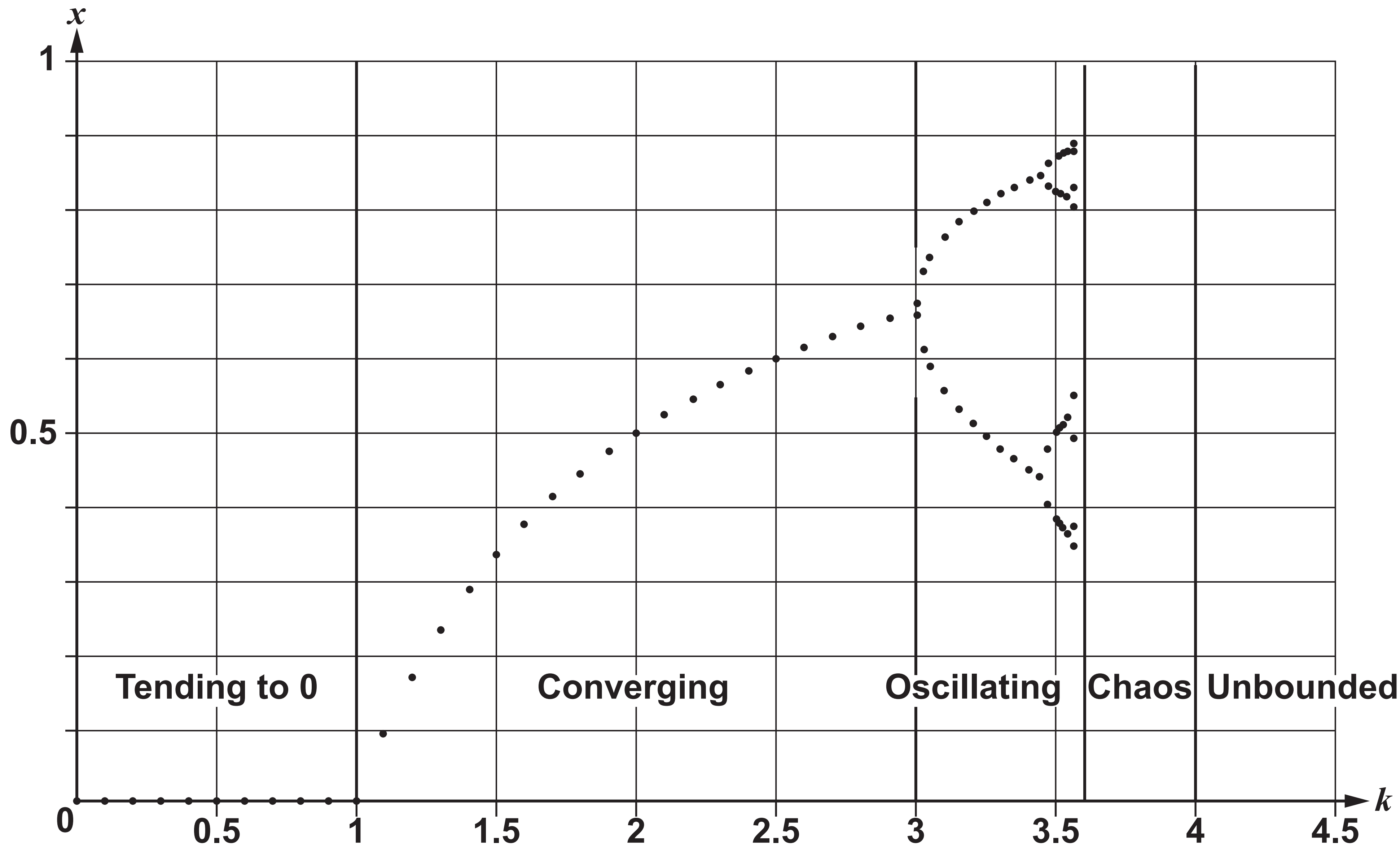
For $0 \leqslant k \leqslant 1$, the iteration converges to zero.

Converging to a single non-zero value

For $1 < k < 3$, the iteration converges to a non-zero value between 0 and 1.

An example of this is given in Table 1 for $k = 1.5$. The iteration is found to converge to $\frac{1}{3}$. This value may be described as an ‘equilibrium point’.

FIG. 2



x
1
0.5
0
k
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Tending to 0
Converging
Oscillating
Chaos
Unbounded

It can be found algebraically. Denoting it by x gives the equation

$$x = 1.5x(1 - x)$$

which has a solution of $x = 0$ or $\frac{1}{3}$.

Oscillating

Table 1 shows that for $k = 3.3$, x_n oscillates between two values. There is a range of values of k for which this occurs.

> For $k = 2.99$, x_n converges slowly to 0.6655... . At $k = 3$, it starts to oscillate. After 5000 iterations the low value is 0.6633... and the high value is 0.6699... . This is a cycle of length 2.

Thus it is found, for example by experiment using a spreadsheet, that the smallest value of k for which the iteration oscillates is 3. Such a value of k where the iteration splits is called a ‘point of bifurcation’.

However, $k = 3$ is not the only point of bifurcation. At $k = 3.449...$ there is a further point of bifurcation at which the cycle of length 2 becomes a cycle of length 4.

> For $k = 3.5$, the four values of x are 0.3828..., 0.8269..., 0.5008... and 0.8749... .

Another point of bifurcation occurs at $k = 3.544...$. At this point the length of the cycle goes up to 8. Further points of bifurcation give cycles of length 16, then 32, then 64 and so on.

Chaos

The pattern of cycles of increasing length does not continue indefinitely as k increases. For larger values of k, for example $k = 3.8$, the outcomes have no pattern, and so the situation is described as ‘chaos’. It is very difficult to distinguish between chaos and a long cycle; a sophisticated computer program is required.

A feature of chaos is that the iteration remains bounded. The values of x_n are always between 0 and 1.

Unbounded outcomes

For values of $k \geqslant 4$, the outcomes cease to be bounded. An example of this occurs in the final column of Table 1, where the value of k is 4.5.

Feigenbaum’s Constant

To summarise, from $k = 1$ to 3 the iteration $x_{n+1} = kx_n(1 - x_n)$ converges to a single non-zero value. There is then a point of bifurcation and this is followed by further points of bifurcation. It is evident from Fig. 2 that the intervals between successive points of bifurcation become progressively shorter.

Information about these intervals is given in Table 3. The numbers in this table have been rounded to 4 decimal places.

TABLE 3

Cycle length	Boundary values of k		Interval	$\frac{\text{Previous interval}}{\text{This interval}}$
1	1	3	2	–
2	3	3.4495	0.4495	4.4494
4	3.4495	3.5441	0.0946	4.7516
8	3.5441	3.5644	0.0203	4.6601
16	3.5644	...	...	...

The right hand column gives the ratio by which the length of this interval decreases with successive cycles. The three values of this ratio in Table 3 are close together.

In the 1970s, Mitchell Feigenbaum started to investigate this ratio. He discovered that similar patterns of bifurcation are found with many other iterations; examples include $x_{n+1} = x_n^2 + k$ and $x_{n+1} = k\sin(\pi x_n)$.

He also discovered that the ratio tends to a definite limit and that this has the same value for all iterations that show this pattern of bifurcation.

His work was conducted to a very high level of accuracy and covered many more cycles than the small number considered here. The limited power of computers in those days meant that it was an enormous undertaking.

Feigenbaum was immediately convinced of the importance of his discovery.

‘I called my parents that evening and told them I had discovered something truly remarkable, that, when I had understood it, would make me a famous man.’

He is indeed now a famous man and the ratio he discovered is called Feigenbaum’s Constant. It has now been found to over 1000 figures; the first ten of these are 4.669 201 609.

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